Differentiation and Integration

1

Find the value of the following question for the function $f(x)=x^2-2$.

(1) Average rate of change when the value of x varies from -2 to 1

(2) Differential coefficient at $x=-1$

(3) The value of t when the slope of the tangent line at point A$(t,\ f(t))$ on the curve $y=f(x)$ is 2

solution

(1) The average rate of change to be sought is $\dfrac{f(1)-f(-2)}{1-(-2)}=\dfrac{(1^2-2)-\{(-2)^2-2\}}{1-(-2)}=\mathbf{-1}$.

(2) The differential coefficient to be sought is

$$f'(-1)=\lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}=\lim_{h\to 0}\frac{\{(-1+h)^2-2\}-\{(-1)^2-2\}}{h}$$

$$=\lim_{h\to 0}\frac{1-2h+h^2-2-1+2}{h}=\lim_{h\to 0}\frac{-2h+h^2}{h}=\lim_{h\to 0}(-2+h)=\mathbf{-2}.$$

(3) $$f'(t)=\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}=\lim_{h\to 0}\frac{\{(t+h)^2-2\}-(t^2-2)}{h}=\lim_{h\to 0}\frac{t^2+2ht+h^2-2-t^2+2}{h}$$

$$=\lim_{h\to 0}\frac{2ht+h^2}{h}=\lim_{h\to 0}(2t+h)=2t$$

Since the slope of the tangent line at point A is 2, $f'(t)=2$.

Therefore, $2t=2$, and thus $\boldsymbol{t=1}$.

2

Differentiate the function $f(x)=x^2+3x$ according to the definition.

solution

$$\boldsymbol{f'(x)} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\{(x+h)^2+3(x+h)\}-(x^2+3x)}{h}$$

$$= \lim_{h\to 0}\frac{x^2+2hx+h^2+3x+3h-x^2-3x}{h} = \lim_{h\to 0}\frac{2hx+h^2+3h}{h} = \lim_{h\to 0}(2x+h+3) = \boldsymbol{2x+3}\,.$$

3

Differentiate the following functions.

(1) $y=x^3-3x^2-3x-6$ (2) $y=(x+2)(x-4)^2$

solution

(1) $\boldsymbol{y'}=(x^3-3x^2-3x-6)'=(x^3)'-3(x^2)'-3(x)'-(6)'=3x^2-3\cdot 2x-3\cdot 1-0=\boldsymbol{3x^2-6x-3}$

(2) $y=(x+2)(x-4)^2=(x+2)(x^2-8x+16)=x^3-8x^2+16x+2x^2-16x+32=x^3-6x^2+32$, so it is

$\boldsymbol{y'}=(x^3-6x^2+32)'=(x^3)'-6(x^2)'+(32)'=3x^2-6\cdot 2x+0=\boldsymbol{3x^2-12x}$.

4

(1) Find the differential coefficient at $x=-3$ for the following function $f(x)$.

① $f(x)=2x^2+4x$ ② $f(x)=x^3+4x^2+x+2$

(2) The position $f(t)$ m of an object moving in a straight line after t seconds is represented by $f(t)=t^2+3t$.

Find the following.

① Average speed from 1 second to 5 seconds later

② Instantaneous speed after 3 seconds

solution

(1) ① $f'(x)=4x+4$, so it is

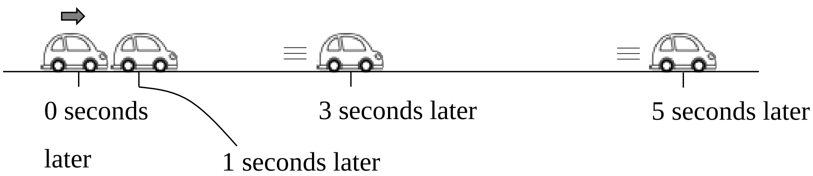
$f'(-3)=4\cdot(-3)+4=\mathbf{-8}$.

② $f'(x)=3x^2+4\cdot 2x+1+0=3x^2+8x+1$, so it is

$f'(-3)=3\cdot(-3)^2+8\cdot(-3)+1=\mathbf{4}$.

(2) ① The average speed to be sought is

$$\frac{5^2+3\cdot 5-(1^2+3\cdot 1)}{5-1}=\mathbf{9\ (m/s)}\ .$$



② $f'(t)=2t+3$.

Therefore, the instantaneous speed we seek is $f'(3)=2\cdot 3+3=\mathbf{9\ (m/s)}$.

5

(1) Find the equation of the tangent line at the point (1, 2) on the curve $y=x^3+x^2$.

(2) Find the equation of the tangent line drawn from point (1, −1) to the curve $y=x^2+2x$.

solution

(1) If $f(x)=x^3+x^2$, then $f'(x)=3x^2+2x$.

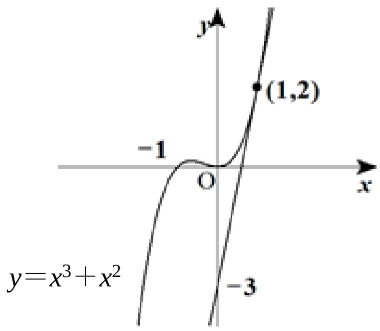
The slope of the tangent line at point (1, 2) is

$$f'(1)=3\cdot 1^2+2\cdot 1=5 .$$

Therefore, the equation of the tangent line to be sought is

$$y-2=5(x-1) .$$

That is, $\boldsymbol{y=5x-3}$.



(2) If $f(x)=x^2+2x$, then $f'(x)=2x+2$.

If the coordinates of the contact point are $(a,\ a^2+2a)$,

the slope of the tangent line at that point is

$$f'(a)=2a+2 .$$

Therefore, the equation of this tangent line is

$$y-(a^2+2a)=(2a+2)(x-a) .$$

That is, $y=(2a+2)x-a^2$. ……①

Since the straight line ① passes through the point (1, −1),

$$-1=(2a+2)\cdot 1-a^2 .$$

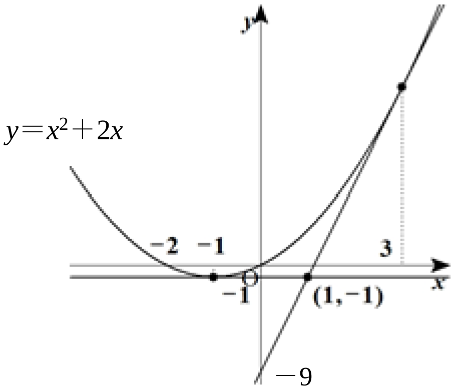
To summarize, $(a+1)(a-3)=0$.

Solving for this, $a=-1,\ 3$.

When $a=-1$, ① is $y=-1$.

When $a=3$, ① is $y=8x-9$.

From the above, the equation of the tangent line we seek is $\boldsymbol{y=-1,\ y=8x-9}$.



6

(1) Find the increase or decrease of the function $y=x^3+3x^2-9x-7$.

(2) Examine and graph the extreme values of the following functions.

① $y=-2x^3+x^2+8x$ ② $y=-3x^3+3x^2-x+1$

solution

(1) $y'=3x^2+6x-9=3(x^2+2x-3)=3(x+3)(x-1)$

If $y'=0$, then $x=-3,\ 1$.

The table of increase/decrease of y is shown on the right.

x	$\cdots$	-3	$\cdots$	1	$\cdots$
y'	$+$	0	$-$	0	$+$
y	↗	20	↘	-12	↗

Thus,

y increases for $x \leqq -3,\ 1 \leqq x$ and decreases for $-3 \leqq x \leqq 1$.

(2) ① $y'=-6x^2+2x+8$

$=-2(3x^2-x-4)$

$=-2(x+1)(3x-4)$

1	1	→	3
3	−4	→	−4
			−1

If $y'=0$, then

$x=-1,\ \dfrac{4}{3}$.

x	$\cdots$	-1	$\cdots$	$\frac{4}{3}$	$\cdots$
y'	$-$	0	$+$	0	$-$
y	↘	-5	↗	$\frac{208}{27}$	↘

The table of increase/decrease of y is shown on the right.

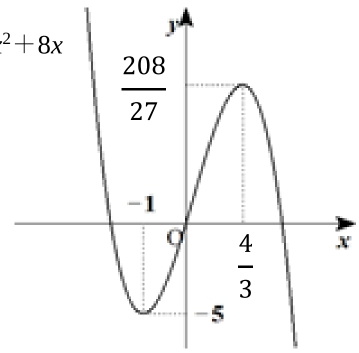
Therefore,

the minimal is at $x=-1$ and the minimal value is -5,

and the maximal is at $x=\dfrac{4}{3}$ and the maximal value is $\dfrac{208}{27}$.

Thus, the graph is shown in the figure on the right.

$y=-2x^3+x^2+8x$



② $y'=-9x^2+6x-1$

$=-(9x^2-6x+1)$

$=-(3x-1)^2$

If $y'=0$, then $x=\dfrac{1}{3}$.

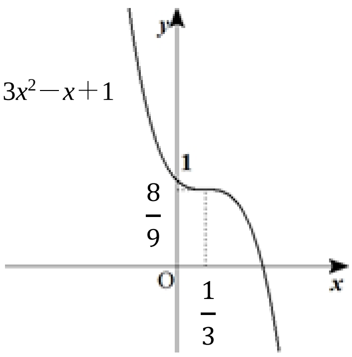
x	$\cdots$	$\frac{1}{3}$	$\cdots$
y'	$-$	0	$-$
y	↘	$\frac{8}{9}$	↘

The table of increase/decrease of y is shown on the right.

Therefore, **it has no extreme value.**

The graph is shown in the figure on the right.

$y=-3x^3+3x^2-x+1$



7

When the cubic function $f(x) = -x^3 + ax^2 + bx + 1$ reaches a minimal at $x = -\frac{1}{3}$ and a maximal at $x = 1$,

find the values of the constants a and b .

solution

$f'(x) = -3x^2 + 2ax + b$. $f'\left(-\frac{1}{3}\right) = 0$, $f'(1) = 0$, since it is extreme at $x = -\frac{1}{3}, x = 1$.

Therefore, $\begin{cases} -\frac{1}{3} - \frac{2}{3}a + b = 0 \\ -3 + 2a + b = 0 \end{cases}$. Solve this and it is $a = 1$, $b = 1$.

At this time, $f(x) = -x^3 + x^2 + x + 1$,

$f'(x) = -3x^2 + 2x + 1 = -(3x^2 - 2x - 1)$

$= -(3x + 1)(x - 1)$.

3	1	→	1
1	− 1	→	− 3
			− 2

If $f'(x) = 0$, then $x = -\frac{1}{3}$, 1,

and so the table of increase/decrease is shown on the right .

x	$\cdots$	$-\frac{1}{3}$	$\cdots$	1	$\cdots$
$f'(x)$	−	0	+	0	−
$f(x)$	↘	$\frac{22}{27}$	↗	2	↘

$f(x)$ is minimal at $x = -\frac{1}{3}$ and maximal at $x = 1$,

thus satisfying the subject .

From the above, $\boldsymbol{a = 1}$, $\boldsymbol{b = 1}$ **.**

8

(1) When the function $f(x)=-x^3+x^2-3ax+2$ has extreme values, find the range of possible values for the constant a.

(2) Find the range of values of the constant a such that the function $f(x)=x^3+2ax^2+3x-4$ has no extreme values.

solution

(1) $f'(x)=-3x^2+2x-3a$.

The condition for $f(x)$ to have extreme values is that $f'(x)=0$ has two different real solutions.

Let D be the discriminant formula for $f'(x)=0$. $D=2^2-4\cdot(-3)\cdot(-3a)=4-36a$.

Therefore, $D>0$ for $\boldsymbol{a<\frac{1}{9}}$.

(2) $f'(x)=3x^2+4ax+3$.

The condition for $f(x)$ not to have extreme values is that $f'(x)=0$ has only 1 real solution or no real solution.

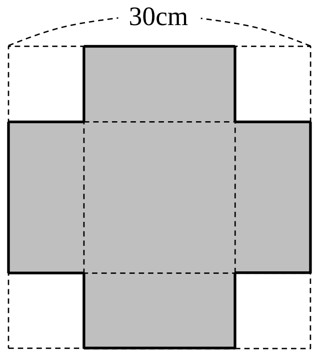
Let D be the discriminant formula for $f'(x)=0$. $D=(4a)^2-4\cdot3\cdot3=4(2a+3)(2a-3)$.

Therefore, $D\leqq0$ for $\boldsymbol{-\frac{3}{2}\leqq a\leqq\frac{3}{2}}$.

9

(1) Find the maximum and minimum values of the function $y=x^3-2x^2$ in the interval $-1 \leqq x \leqq 2$.

(2) Cut out squares of the same size from the four corners of a square of cardboard with 1 side of 30 cm to make a rectangular box without a lid. In this case, to maximize the volume of the box, how many centimeters should 1 side of the square to be cut out be?



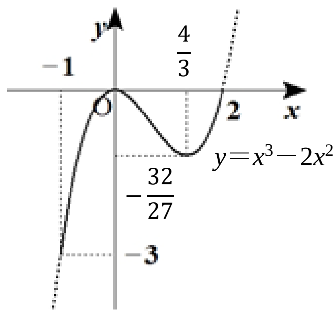
solution

(1) $y'=3x^2-4x$

$\quad =x(3x-4)$.

If $y'=0$, then

$x=0,\ \dfrac{4}{3}$.



The table of increase/decrease of y is shown on the right.

Thus,

the maximum value 0 at $x=0$, 2 and

the minimum value -3 at $x=-1$.

x	-1	$\cdots$	0	$\cdots$	$\frac{4}{3}$	$\cdots$	2
y'		$+$	0	$-$	0	$+$	
y	-3	↗	0	↘	$-\frac{32}{27}$	↗	0

(2) 1 side of the square to be cut out is x cm .

$x>0,\ 30-2x>0$ to $0<x<15$.

If the volume of the box is V(cm^3), then

$V=x(30-2x)^2=x(900-120x+4x^2)$

$\quad =4x^3-120x^2+900x$,

$V'=12x^2-240x+900=12(x^2-20x+75)$

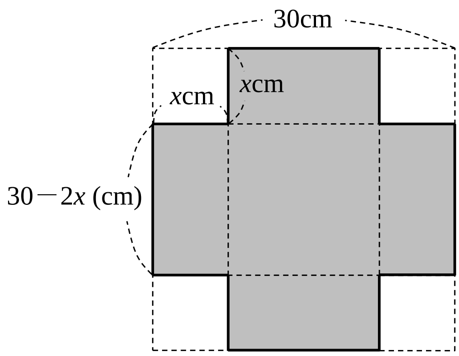
$\quad =12(x-5)(x-15)$.

For $0<x<15$, $V'=0$ at $x=5$.

The table of increase/decrease in V is shown on the right.

Therefore, the maximum value of 2000 is taken at $x=5$.

Thus, 1 side of the square to be cut out should be **5 cm** .



x	0	$\cdots$	5	$\cdots$	15
V'		$+$	0	$-$	
V		↗	2000	↘	

10

When the cubic equation $2x^3-6x+a=0$ has 3 different real solutions, find the range of possible values of the constant a .

solution

Transforming the equation, we obtain $-2x^3+6x=a$ ……① .

Let $\begin{cases} y=-2x^3+6x & \cdots\cdots② \\ y=a & \cdots\cdots③ \end{cases}$.

Then the number of different real solutions of equation ① corresponds to the number of shared points in the graphs of ② and ③ .

At $y=-2x^3+6x$,

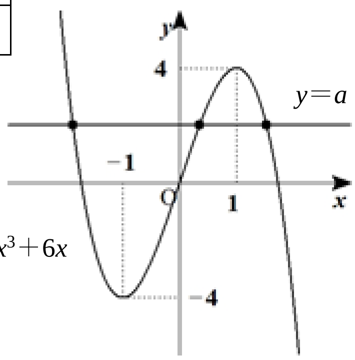
$y'=-6x^2+6=-6(x+1)(x-1)$.

If $y'=0$, then $x=-1$, 1 .

The table of increase/decrease and the graphs are shown on the right.

x	$\cdots$	-1	$\cdots$	1	$\cdots$
y'	$-$	0	$+$	0	$-$
y	↘	-4	↗	4	↘

The range of values of a to be sought is $\boldsymbol{-4<a<4}$, since the graph of $y=-2x^3+6x$ and the line $y=a$ have 3 common points.



$y=-2x^3+6x$

Alternative solution

The real solution of the equation $f(x)=0$ is the x-coordinate of the common point of the graph of $y=f(x)$ and the x-axis. From the signs of the extreme values, we can also find the range of values of a that satisfy the subject.

If $y=2x^3-6x+a$, then

$y'=6x^2-6=6(x+1)(x-1)$.

If $y'=0$, then $x=-1$, 1 .

The table of increase/decrease of y is shown on the right.

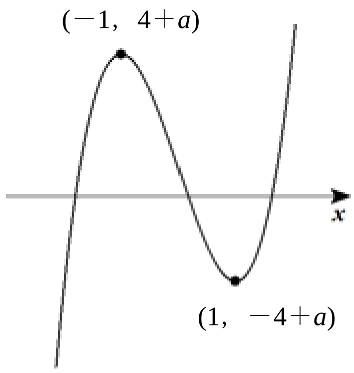
x	$\cdots$	-1	$\cdots$	1	$\cdots$
y'	$+$	0	$-$	0	$+$
y	↗	$4+a$	↘	$-4+a$	↗

From the positional relationship between the extreme values and the x-axis, the subject is satisfied if

$$\begin{cases} 4+a>0 \\ -4+a<0 \end{cases}.$$

Therefore, $-4<a<4$.

〈Note〉 Since the y-coordinates of the two extreme values should have different signs, $(4+a)(-4+a)<0$ may be used.



$y=2x^3-6x^2+a$

11

Prove that the inequality $x^3+80 \geqq 3x(x+8)$ is satisfied when $x \geqq 0$.

proof

Let $f(x)=x^3+80-3x(x+8)=x^3-3x^2-24x+80$.

$f'(x)=3x^2-6x-24=3(x^2-2x-8)=3(x+2)(x-4)$.

If $f'(x)=0$, then $x=-2,\ 4$.

The table of increase/decrease of $f(x)$ at $x \geqq 0$ is shown on the right.

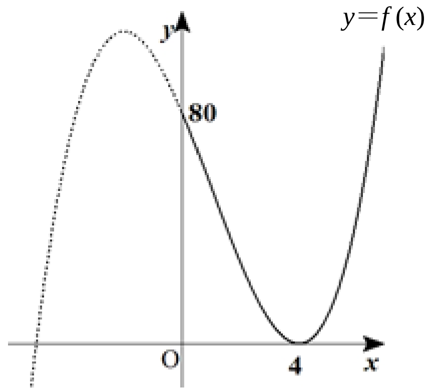
x	0	$\cdots$	4	$\cdots$
$f'(x)$		$-$	0	$+$
$f(x)$	80	↘	0	↗

Therefore, $f(x)$ takes the minimum value 0 at $x=4$ when $x \geqq 0$.

Thus, $f(x) \geqq 0$.

That is, $x^3+80-3x(x+8) \geqq 0$.

From the above, $x^3+80 \geqq 3x(x+8)$ when $x \geqq 0$.



12

(1) Find the following indefinite integrals.

① $\int (2x+1)\,dx$ ② $\int (x^2-3x-5)\,dx$ ③ $\int (2t^2+1)(2t-3)\,dt$

(2) Find a function $f(x)$ satisfying $f'(x)=3x^2-x$, $f(2)=7$.

solution

Let C be the integration constant.

(1) ① $\int (2x+1)\,dx = 2\int x\,dx + \int dx = 2\cdot\frac{1}{2}x^2 + x + C = \boldsymbol{x^2+x+C}$

② $\int (x^2-3x-5)\,dx = \int x^2\,dx - 3\int x\,dx - 5\int dx = \frac{1}{3}x^3 - 3\cdot\frac{1}{2}x^2 - 5x + C$

$$= \boldsymbol{-\frac{1}{3}x^3 - \frac{3}{2}x^2 - 5x + C}$$

③ $\int (2t^2+1)(2t-3)\,dt = \int (4t^3-6t^2+2t-3)\,dt = 4\int t^3\,dt - 6\int t^2\,dt + 2\int t\,dt - 3\int dt$

$$= 4\cdot\frac{1}{4}t^4 - 6\cdot\frac{1}{3}t^3 + 2\cdot\frac{1}{2}t^2 - 3t + C = \boldsymbol{t^4 - 2t^3 + t^2 - 3t + C}$$

(2) Since $f(x)$ is a primitive function of $3x^2-x$, $f(x) = \int (3x^2-x)\,dx = x^3 - \frac{1}{2}x^2 + C$.

Where $f(2) = 2^3 - \frac{1}{2}\cdot 2^2 + C = 6 + C$. Since $f(2) = 7$, then $6 + C = 7$.

Therefore, $C = 1$. Thus, $\boldsymbol{f(x) = x^3 - \frac{1}{2}x^2 + 1}$.

13

(1) Find the following definite integrals.

① $\int_{-1}^{1}(x^2-3)\,dx$　　② $\int_{0}^{2}(2t+1)(4t^2-2t+1)\,dt$

③ $\int_{1}^{3}x^2(x-4)\,dx+4\int_{1}^{3}x(x-1)\,dx-\int_{2}^{3}x(x+2)(x-2)\,dx$

(2) Find a function $f(x)$ satisfying the equality $f(x)=2x^2+2x-\int_{-3}^{0}f(t)\,dt$.

solution

(1) ① $\int_{-1}^{1}(x^2-3)\,dx=\left[\frac{1}{3}x^3-3x\right]_{-1}^{1}=\left(\frac{1}{3}-3\right)-\left(-\frac{1}{3}+3\right)=-\mathbf{\frac{16}{3}}$

② $\int_{0}^{2}(2t+1)(4t^2-2t+1)\,dt=\int_{0}^{2}(8t^3+1)\,dt=\left[8\cdot\frac{1}{4}t^4+t\right]_{0}^{2}=\left[2t^4+t\right]_{0}^{2}=(32+2)-0=\mathbf{34}$

③ $\int_{1}^{3}x^2(x-4)\,dx+4\int_{1}^{3}x(x-1)\,dx-\int_{2}^{3}x(x+2)(x-2)\,dx$

$=\int_{1}^{3}\{x^2(x-4)+4x(x-1)\}\,dx-\int_{2}^{3}x(x^2-4)\,dx$

$=\int_{1}^{3}(x^3-4x^2+4x^2-4x)\,dx-\int_{2}^{3}(x^3-4x)\,dx=\int_{1}^{3}(x^3-4x)\,dx+\int_{3}^{2}(x^3-4x)\,dx$

$=\int_{1}^{2}(x^3-4x)\,dx=\left[\frac{1}{4}x^4-4\cdot\frac{1}{2}x^2\right]_{1}^{2}=\left[\frac{1}{4}x^4-2x^2\right]_{1}^{2}$

$=(4-8)-\left(\frac{1}{4}-2\right)=-\mathbf{\frac{9}{4}}$

(2) Since $\int_{-3}^{0}f(t)\,dt$ is a constant, let $\int_{-3}^{0}f(t)\,dt=a$.

In this case, $f(x)=2x^2+2x-a$.

Therefore, $\int_{-3}^{0}f(t)\,dt=\int_{-3}^{0}(2t^2+2t-a)\,dt=\left[2\cdot\frac{1}{3}t^3+2\cdot\frac{1}{2}t^2-at\right]_{-3}^{0}=\left[\frac{2}{3}t^3+t^2-at\right]_{-3}^{0}$

$=0-(-18+9+3a)=9-3a$.

$\int_{-3}^{0}f(t)\,dt=a$, so it is $9-3a=a$. Solve this and it is $a=\frac{9}{4}$.

Thus, $\boldsymbol{f(x)=2x^2+2x-\frac{9}{4}}$.

14

Find the function $f(x)$ satisfying the equality $\int_a^x f(t)\,dt = 3x^2 + 4x + 1$

and the value of the constant a, respectively.

solution

Differentiating both sides of the equation with respect to x, $\boldsymbol{f(x)=6x+4}$.

Also, in the given equality, if $x = a$, then $\int_a^a f(t)\,dt = 3a^2 + 4a + 1$.

Since the left-hand side is zero, $0=3a^2+4a+1$.

Therefore, $(a+1)(3a+1)=0$.

1	1	→	3
3	1	→	1
			4

Thus, $\boldsymbol{a = -1,\ -\frac{1}{3}}$.

15

Find the area S of the figures bounded by the following curves and lines.

(1) $y=2x^2+2x$, x-axis, $x=1$, $x=2$

(2) $y=-x^2+4$, x-axis

(3) $y=x^2-3x+2$, x-axis

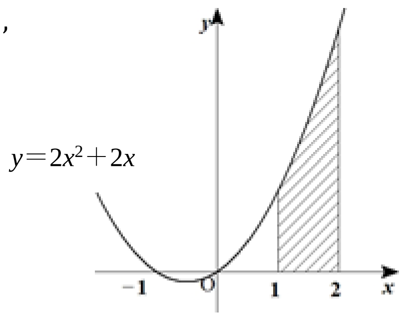
(4) $y=x^2+2x+3$, $y=-2x$

(5) $y=(x+1)^2$, $y=-x^2+5$

solution

(1) From $y=2x^2+2x=2(x^2+x)=2\left\{\left(x+\frac{1}{2}\right)^2-\frac{1}{4}\right\}=2\left(x+\frac{1}{2}\right)^2-\frac{1}{2}$,

$y \geqq 0$ for $1 \leqq x \leqq 2$ from the figure on the right, the area S to be found

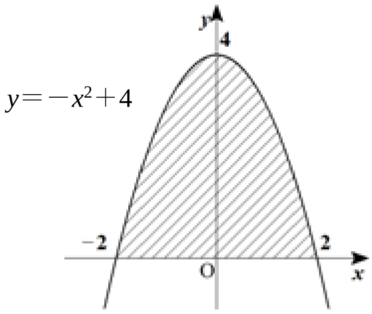


$$S=\int_1^2 (2x^2+2x)\,dx$$

$$=\left[\frac{2}{3}x^3+x^2\right]_1^2=\left(\frac{16}{3}+4\right)-\left(\frac{2}{3}+1\right)=\mathbf{\frac{23}{3}}.$$

(2) $y=-x^2+4=-(x+2)(x-2)$.

Since $y \geqq 0$ in the interval $-2 \leqq x \leqq 2$, the area S to be found is



$$S=\int_{-2}^2 (-x^2+4)\,dx$$

$$=\left[-\frac{1}{3}x^3+4x\right]_{-2}^2=\left(-\frac{8}{3}+8\right)-\left(\frac{8}{3}-8\right)=\mathbf{\frac{32}{3}}.$$

Alternative solution

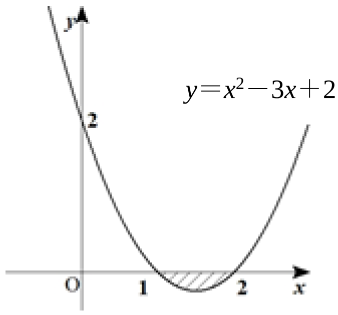
Use the formula $\displaystyle\int_\alpha^\beta (x-\alpha)(x-\beta)\,dx=-\frac{1}{6}(\beta-\alpha)^3$.

$\left(\text{Same as the solution until } S=\displaystyle\int_{-2}^2(-x^2+4)\,dx\,.\right)$

$$S=\int_{-2}^2(-x^2+4)\,dx=-\int_{-2}^2(x+2)(x-2)\,dx=-\left[-\frac{1}{6}\{2-(-2)\}^3\right]=\frac{32}{3}.$$

(3) $y=x^2-3x+2=(x-1)(x-2)$.

Since $y \leqq 0$ in the interval $1 \leqq x \leqq 2$, the area S to be found is



$$S=-\int_1^2 (x^2-3x+2)\,dx$$

$$=-\left[\frac{1}{3}x^3-\frac{3}{2}x^2+2x\right]_1^2=-\left(\frac{8}{3}-6+4\right)+\left(\frac{1}{3}-\frac{3}{2}+2\right)=\mathbf{\frac{1}{6}}.$$

Alternative solution

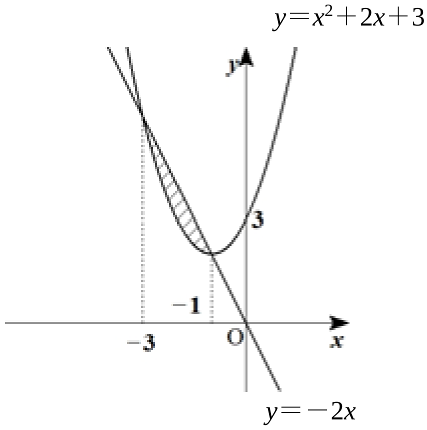
Use the formula $\displaystyle\int_{\alpha}^{\beta}(x-\alpha)(x-\beta)\,dx=-\frac{1}{6}(\beta-\alpha)^3$.

$\left(\text{Same as the solution until } S=-\displaystyle\int_1^2(x^2-3x+2)\,dx\,.\right)$

$$S=-\int_1^2(x^2-3x+2)\,dx=-\int_1^2(x-1)(x-2)\,dx=-\left\{-\frac{1}{6}(2-1)^3\right\}=\frac{1}{6}.$$

(4) Solving $x^2+2x+3=-2x$, $x=-3,\ -1$, since $(x+3)(x+1)=0$.

Since $-2x\geqq x^2+2x+3$ in the interval $-3\leqq x\leqq -1$, the area S to be found is



$$S=\int_{-3}^{-1}\{-2x-(x^2+2x+3)\}\,dx=\int_{-3}^{-1}(-x^2-4x-3)\,dx$$

$$=\left[-\frac{1}{3}x^3-2x^2-3x\right]_{-3}^{-1}=\left(\frac{1}{3}-2+3\right)-(9-18+9)$$

$$=\mathbf{\frac{4}{3}}.$$

Alternative solution

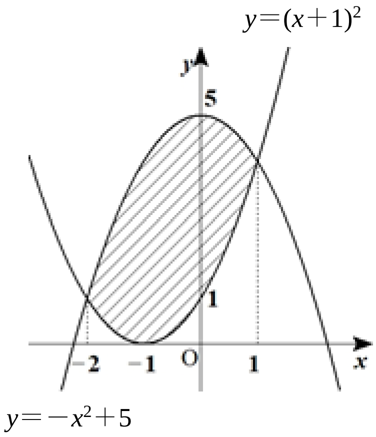
Use the formula $\displaystyle\int_{\alpha}^{\beta}(x-\alpha)(x-\beta)\,dx=-\frac{1}{6}(\beta-\alpha)^3$.

$\left(\text{Same as the solution until } S=\displaystyle\int_{-3}^{-1}(-x^2-4x-3)\,dx\,.\right)$

$$S=\int_{-3}^{-1}(-x^2-4x-3)\,dx=-\int_{-3}^{-1}(x+3)(x+1)\,dx=-\left[-\frac{1}{6}\{-1-(-3)\}^3\right]=\frac{4}{3}.$$

(5) Solving $(x+1)^2=-x^2+5$, $x=-2,\ 1$, since $2x^2+2x-4=0$, $2(x+2)(x-1)=0$.

Since $-x^2+5\geqq(x+1)^2$ in the interval $-2\leqq x\leqq 1$, the area S to be found is



$$S=\int_{-2}^{1}\{(-x^2+5)-(x+1)^2\}\,dx=\int_{-2}^{1}(-2x^2-2x+4)\,dx$$

$$=\left[-\frac{2}{3}x^3-x^2+4x\right]_{-2}^{1}=\left(-\frac{2}{3}-1+4\right)-\left(\frac{16}{3}-4-8\right)$$

$$=\mathbf{9}.$$

Alternative solution

Use the formula $\displaystyle\int_{\alpha}^{\beta}(x-\alpha)(x-\beta)\,dx=-\frac{1}{6}(\beta-\alpha)^3$.

$\left(\text{Same as the solution until } S=\displaystyle\int_{-2}^{1}(-2x^2-2x+4)\,dx\,.\right)$

$$S=\int_{-2}^{1}(-2x^2-2x+4)\,dx=-2\int_{-2}^{1}(x+2)(x-1)\,dx=-2\left[-\frac{1}{6}\{1-(-2)\}^3\right]=9.$$

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(1) Find the area S of the figure bounded by the curve $y=x^3-7x+6$ and the x-axis.

(2) ① Find the equation of the tangent line drawn from point (2, 3) to the curve $y=x^2$.

② Find the area S of the figure bounded by the two tangent lines and the curve $y=x^2$ obtained in ①.

solution

(1) If $f(x)=x^3-7x+6$,

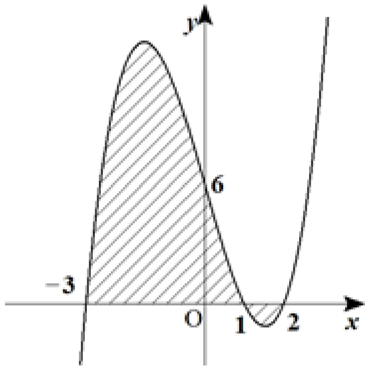
then $f(x)=(x-1)(x^2+x-6)$

$=(x-1)(x+3)(x-2)$

from $f(1)=1-7+6=0$.

1	1	0	−7	6
		1	1	−6
	1	1	−6	0

$y=x^3-7x+6$



Therefore, the intersections of the curve y with the x-axis are (−3, 0), (1, 0), and (2, 0), so the approximate shape of the curve is shown in the figure on the right, and the area S is the area of the figure in the shaded area.

$$S=\int_{-3}^{1}(x^3-7x+6)\,dx+\int_{1}^{2}\{-(x^3-7x+6)\}\,dx$$

$$=\left[\frac{1}{4}x^4-\frac{7}{2}x^2+6x\right]_{-3}^{1}+\left[-\frac{1}{4}x^4+\frac{7}{2}x^2-6x\right]_{1}^{2}$$

$$=\left(\frac{1}{4}-\frac{7}{2}+6\right)-\left(\frac{81}{4}-\frac{63}{2}-18\right)+(-4+14-12)-\left(-\frac{1}{4}+\frac{7}{2}-6\right)$$

$$=\boldsymbol{\frac{131}{4}}.$$

(2) ① If $f(x)=x^2$, then $f'(x)=2x$.

Let the coordinates of the contact point be $(a,\ a^2)$,

the slope of the tangent line at that point is $f'(a)=2a$.

Therefore, the equation of this tangent line is

$y-a^2=2a(x-a)$.

That is, $y=2ax-a^2$ ……① .

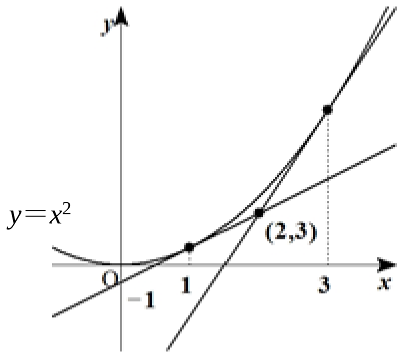
Since straight line ① passes through point (2, 3),

$3=4a-a^2$, $a^2-4a+3=0$, $(a-1)(a-3)=0$.

Solving for this, $a=1$, 3 .

When $a=1$, ① is $y=2x-1$.

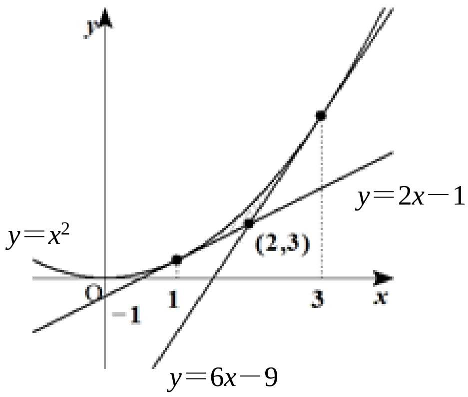
When $a=3$, ① is $y=6x-9$.



From the above, the equations of the tangent line to be obtained are

$\boldsymbol{y=2x-1,\ y=6x-9}$.

② The area S to be found is the area of the shaded area in the figure on the right, so



$$S=\int_1^2\{x^2-(2x-1)\}\,dx+\int_2^3\{x^2-(6x-9)\}\,dx$$

$$=\int_1^2(x^2-2x+1)\,dx+\int_2^3(x^2-6x+9)\,dx$$

$$=\left[\frac{1}{3}x^3-x^2+x\right]_1^2+\left[\frac{1}{3}x^3-3x^2+9x\right]_2^3$$

$$=\left(\frac{8}{3}-4+2\right)-\left(\frac{1}{3}-1+1\right)+(9-27+27)-\left(\frac{8}{3}-12+18\right)=\mathbf{\frac{2}{3}}.$$

Alternative solution

Use the formula $\int(\boldsymbol{x}-\boldsymbol{\alpha})^2\,\boldsymbol{dx}=\frac{(\boldsymbol{x}-\boldsymbol{\alpha})^3}{\mathbf{3}}+\boldsymbol{C}$ (C is the integration constant).

(Same as the solution until $S=\int_1^2(x^2-2x+1)\,dx+\int_2^3(x^2-6x+9)\,dx$.)

$$S=\int_1^2(x^2-2x+1)\,dx+\int_2^3(x^2-6x+9)\,dx=\int_1^2(x-1)^2\,dx+\int_2^3(x-3)^2\,dx$$

$$=\left[\frac{1}{3}(x-1)^3\right]_1^2+\left[\frac{1}{3}(x-3)^3\right]_2^3=\frac{1}{3}-\left(-\frac{1}{3}\right)=\frac{2}{3}.$$