Figure Properties

1

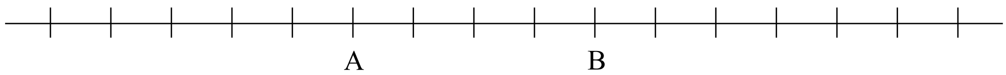
Figure the following points for the line segment AB.

(1) Point P that divides inside into 1 : 3

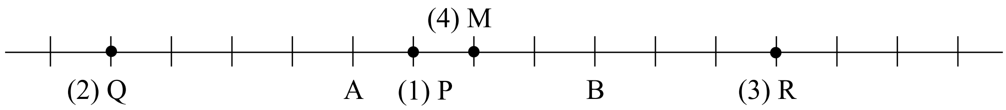
(2) Point Q that divides outside to 1 : 2

(3) Point R that divides outside to 7 : 3

(4) Midpoint M



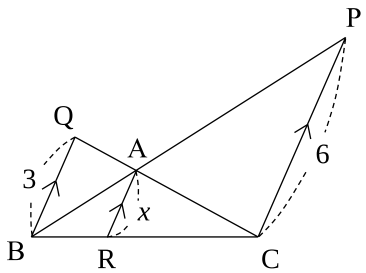
solution



2

In the figure on the right, find the length x of the line segment.

However, AR//BQ, AR//CP, BQ = 3 and CP = 6.



solution

Since △ABQ∽△APC, BA : BP = 1 : 3 from BA : PA = 3 : 6 = 1 : 2 .

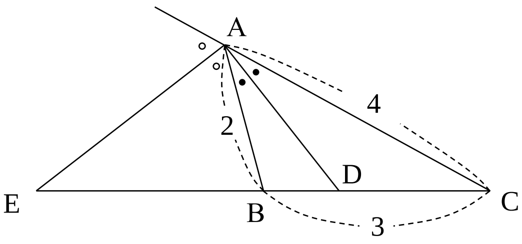
Hence, since AR : CP = 1 : 3, x : 6 = 1 : 3 .

Therefore, x = **2** .

3

In △ABC where AB=2, BC=3, and CA=4, D is the intersection of the bisector of angle A with side BC, and E is the intersection of the bisector of the exterior angle of angle A with the extension of side BC.

Find the length of line segment DE.

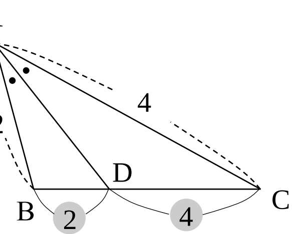


solution

Line segment AD is the bisector of ∠BAC, so

BD：DC＝AB：AC＝2：4＝1：2 .

Therefore,　$BD=\frac{1}{3}BC=\frac{1}{3}\times 3=1$.



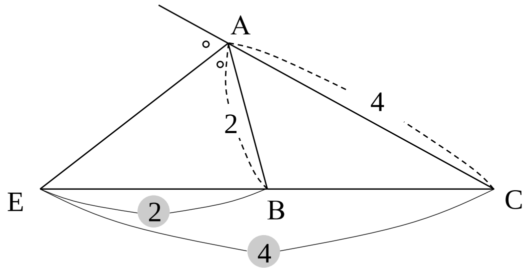
Line AE is the bisector of the exterior angle of ∠BAC, so

BE：EC＝AB：AC＝2：4＝1：2 .

Therefore,　EB：BC＝1：1 .

Thus,　EB＝BC＝3 .

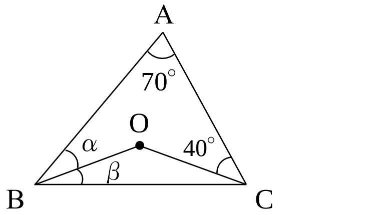
From the above,　DE＝BD＋EB＝1＋3＝**4** .



4

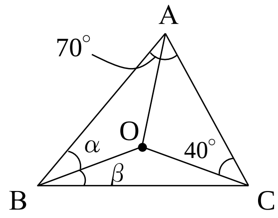
In the figure on the right, point O is the outer center of △ABC.

Find α, β.



solution

Connect 2 points A and O as shown in the figure on the right.



From ∠OCA = ∠OAC, ∠OAB = ∠A − ∠OAC = 70° − 40° = 30°.

From ∠OAB = ∠OBA, **α = 30°**.

∠OBC = ∠OCB, ∠OCA + ∠A + ∠OBA + ∠OBC + ∠OCB = 180°, then

$$40° + 70° + 30° + \beta + \beta = 180°.$$

Therefore, **β = 20°**.

Alternative solution for β

By the circumferential angle theorem, ∠BOC = 2 × ∠A = 2 × 70° = 140°.

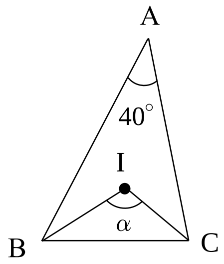
∠OBC = ∠OCB, ∠OBC + ∠OCB + ∠BOC = 180°, then

$$\beta + \beta + 140° = 180°.$$

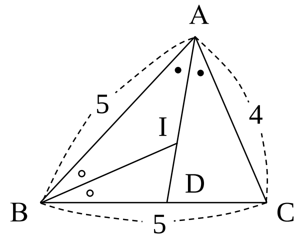
Therefore, β = 20°.

5

(1) In the figure on the right, point I is the interior center of $\triangle$ABC.
Find α .



(2) In $\triangle$ABC where AB=5, BC=5, and CA=4,
let I be the interior center and D be the intersection of line AI and side BC, then find AI : ID.



solution

(1) Let $\angle$IBA$=\angle$IBC$=x$, $\angle$ICA$=\angle$ICB$=y$.

From $\angle A+\angle B+\angle C=180°$,

$40°+2x+2y=180°$.

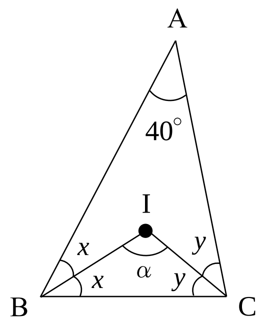
$40°+2(x+y)=180°$

Therefore, $x+y=70°$.

In $\triangle$IBC, $\angle BIC+\angle IBC+\angle ICB=180°$, i.e.,

$\alpha+x+y=180°$.

Thus, $\boldsymbol{\alpha=110°}$.



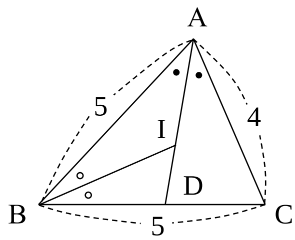
(2) Since straight line AI is the bisector of angle A,

BD : DC＝AB : AC＝5 : 4 .

Therefore, $BD=\frac{5}{9}BC=\frac{5}{9}\times5=\frac{25}{9}$.

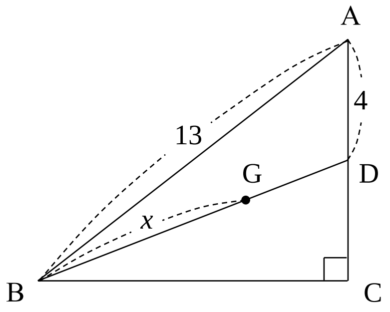
Also, since straight line BI is the bisector of angle B,

AI : ID＝BA : BD＝5 : $\frac{25}{9}$ **＝9 : 5** .



6

In the figure on the right,
point G is the center of gravity of △ABC and ∠C＝90°.
Find the length x of the line segment.



solution

From AD＝DC,　DC＝4,　AC＝8 .

In △ABC, by the Pythagorean theorem,　$BC=\sqrt{13^2-8^2}=\sqrt{105}$.

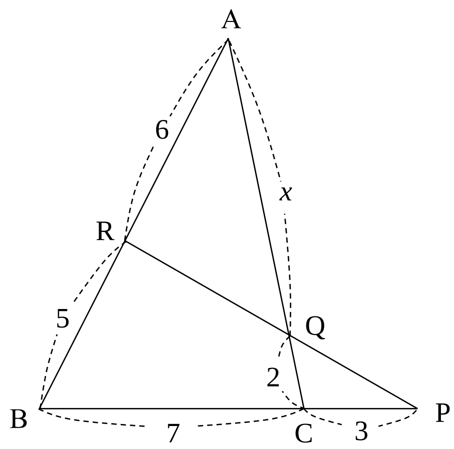
Also, in △DBC,　$BD=\sqrt{(\sqrt{105})^2+4^2}=11$　by Pythagorean theorem.

Since　BG : GD＝2 : 1,　$BG=\frac{2}{3}BD$.

Therefore,　$x=\frac{2}{3}\times 11=\boldsymbol{\frac{22}{3}}$.

7

In the figure on the right, find the length x of the line segment.



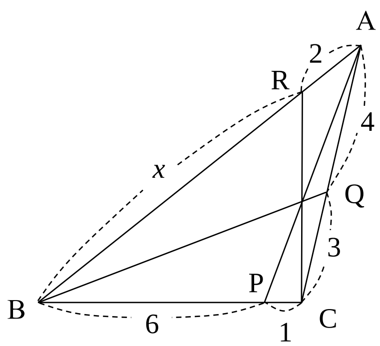
solution

By Menelaus' theorem, $\frac{7+3}{3} \cdot \frac{2}{x} \cdot \frac{6}{5} = 1$.

Therefore, $\boldsymbol{x = 8}$.

8

In the figure on the right, find the length x of the line segment.



solution

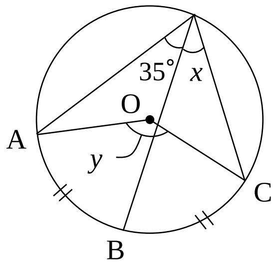
By Ceva's theorem, $\frac{6}{1} \cdot \frac{3}{4} \cdot \frac{2}{x} = 1$.

Therefore, $\boldsymbol{x = 9}$.

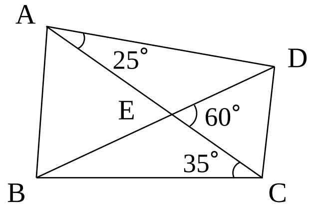
9

(1) In the figure on the right, find x, y.

However, point O is the center of the circle and $\overset{\frown}{AB}=\overset{\frown}{BC}$.



(2) In the figure on the right, are the 4 points A, B, C, and D on the same circumference?



solution

(1) The sizes of the circumferential angles for the same arc are equal, so $\boldsymbol{x=35°}$.

The size of the central angle to an arc is twice the circumferential angle to that arc, so

$y=2\times(35° +35°)=\mathbf{140°}$.

(2) Based on the relationship between the inner and outer angles of △ADE, ∠ADE＝60° －25° ＝35°, so

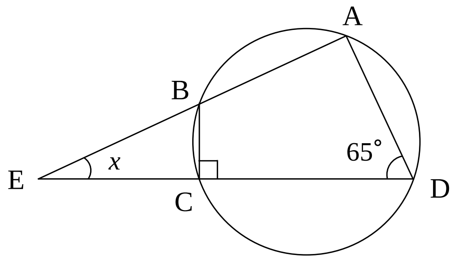
∠ACB＝∠ADB .

Therefore, by the converse of the circumferential angle theorem,

the 4 points A, B, C, and D are on the same circumference.

１０

In the figure on the right, find x .

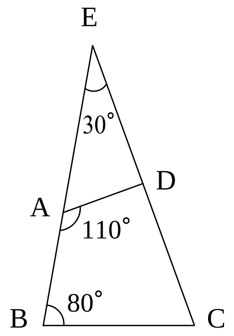


solution

Since ∠BCD＝∠BAD＝90° , $\boldsymbol{x=25°}$ at △AED from 90° ＋65° ＋x＝180° .

１１

Is the right quadrilateral ABCD inscribed in a circle?



solution

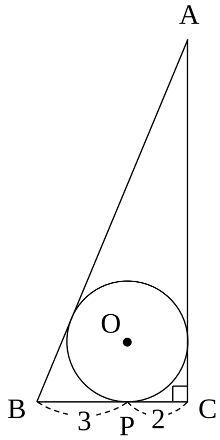
Based on the relationship between the inner and outer angles of △EAD, ∠ADE＝110° －30° ＝80° .

Therefore, since 1 exterior angle is equal to the diagonal of its adjacent interior angle, **quadrilateral ABCD is inscribed in a circle.**

１２

In the figure on the right, circle O is the inscribed circle of right triangle ABC with angle C = 90° and point P is the contact point between side BC and circle O.

Find the lengths of sides AB and AC when BP＝3 and CP＝2.



solution

Let Q and R be the points of contact between circle O and sides AC and AB, respectively,

BP＝BR＝3 and

CP＝CQ＝2 .

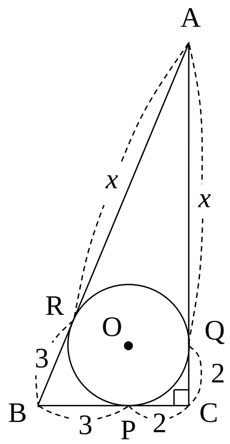
Let　AQ＝AR＝x .

In △ABC, by Pythagorean theorem,

$(x+2)^2+5^2=(x+3)^2$,　　　$x^2+4x+4+25=x^2+6x+9$,

$-2x=-20$,　　　$x=10$.

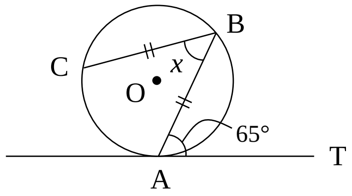
Thus,　**AB＝13,　AC＝12** .



１３

In the figure on the right, the line AT is tangent to the circle O at the point A.

Find x.



solution

Connect 2 points A and C.

In this case, by the theorem of the angle made by the tangent of the circle and the string, ∠BCA＝65°.

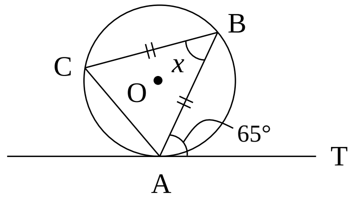
Also, since BC＝BA, ∠BAC＝65°.

Therefore, at △ABC,

∠BCA＋∠BAC＋∠ABC＝180°.

That is, 65°＋65°＋x＝180°.

Thus, **x＝50°**.

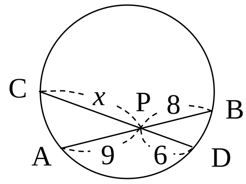


１４

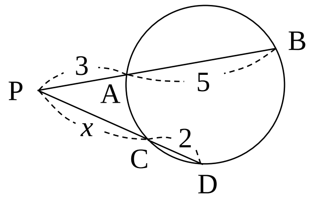
Find the value of x in the following figure.

Note that the line PT in (3) is tangent to the circle whose tangent point is T.

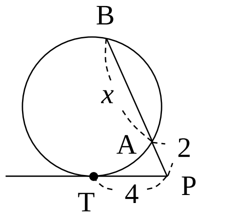
(1)



(2)



(3)



solution

(1) From PA・PB＝PC・PD, $9 \cdot 8 = x \cdot 6$.

Therefore, $\boldsymbol{x=12}$.

(2) From PA・PB＝PC・PD, $3 \cdot (3+5) = x \cdot (x+2)$.

Therefore, $x^2+2x-24=0$, $(x-4)(x+6)=0$.

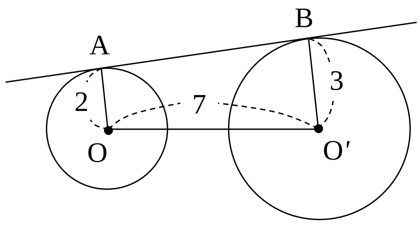
Since $x>0$, $\boldsymbol{x=4}$.

(3) From PA・PB＝PT^2, $2(2+x)=4^2$.

Therefore, $\boldsymbol{x=6}$.

１５

In the figure on the right, straight line AB is the common tangent of the 2 circles O and O′, and points A and B are the points of contact. Find the length of line segment AB.



solution

Draw a perpendicular line OH from point O to line segment O′B .

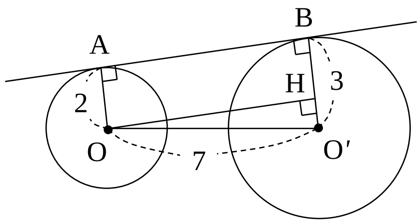
Since OA⊥AB and O′B⊥AB, quadrilateral AOHB is a rectangle.

Therefore, AB＝OH, OA＝HB .

Also, O′H＝O′B－HB＝3－2＝1 .

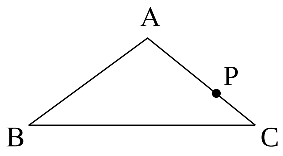
In right triangle OO′H, $OH=\sqrt{OO'^2-O'H^2}=\sqrt{7^2-1^2}=4\sqrt{3}$ by Pythagorean theorem .

Thus, **AB**＝OH＝$\mathbf{4\sqrt{3}}$.

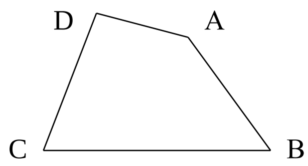


１６

(1) △ABC and a point P on side AC are given.
Draw a line passing through point P and bisecting the area of △ABC.

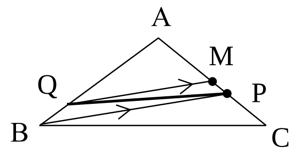


(2) There is a quadrilateral ABCD as shown in the figure on the right.
Draw a line passing through vertex A and bisecting the area of quadrilateral ABCD.

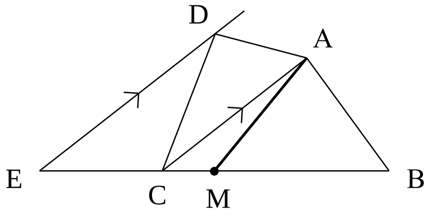


construction

(1) ① Take the midpoint M of the side AC.
② Let Q be the intersection of a line passing through point M and parallel to line BP with side AB.
③ The straight line PQ connecting point P and point Q is the straight line to be sought.

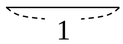


(2) ① Let E be the intersection of a line passing through vertex D and parallel to line AC and line BC.
② Let M be the midpoint of line segment BE.
③ The straight line AM connecting vertex A and point M is the straight line to be sought.



17

Given a line segment of length 1, construct a line segment of length $\sqrt{3}$.

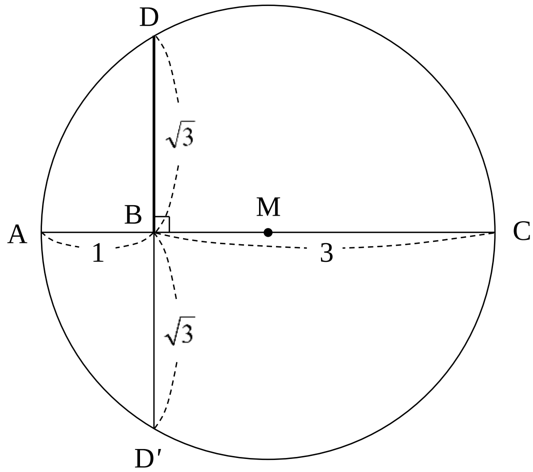


construction

① Take 3 points A, B, and C on the same line such that AB = 1 and BC = 3, in that order.

② Let M be the midpoint of the line segment AC, and draw a circle of radius AM centered at M.

③ Let D and D′ be the intersections of the straight line passing through point B and perpendicular to the straight line AC and the circle drawn in ②.
In this case, the lengths of line segments BD and BD′ are $\sqrt{3}$, since AB・BC = BD・BD′, BD = BD′.

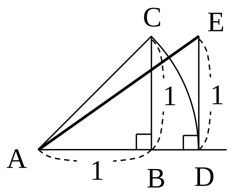


Alternative construction

① Draw a right-angled isosceles triangle ABC with AB＝BC＝1 and angle ABC＝90° .

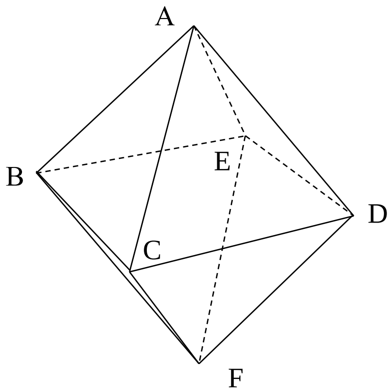
② Take a point D on the half-line AB such that AC＝AD .

③ Draw a right triangle ADE with DE = 1 and angle ADE = 90°.
In this case, since AD = $\sqrt{2}$, the length of line AE is $\sqrt{3}$.



１８

Find the angle between 2 lines AB and EF
in the regular octahedron ABCDEF whose length of one side is 1,
as shown in the figure on the right.

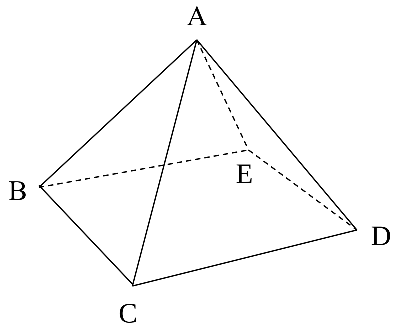


solution

From AC//EF,
the angle between the 2 lines AB and EF is the same as the angle between the 2 lines AB and AC.
Therefore, the angle to be sought is **60°** .

１９

In a regular quadrilateral pyramid A-BCDE with side length 1 as shown in the figure on the right, find the value of $\cos\theta$ when the angle between line AB and plane BCDE is θ.



solution

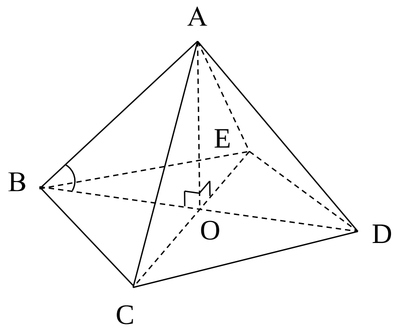
Line AB intersects the plane BCDE at point B.

Let O be the intersection of diagonals BD and CE,

$AO \perp BD$,　$AO \perp CE$.

Therefore, the perpendicular line drawn from point A to plane BCDE is AO, so

$\theta = \angle ABO$.



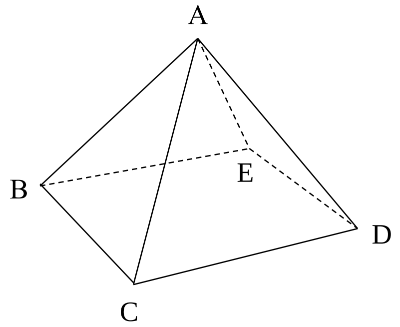
Here,　$AB = 1$,　$BO = \frac{1}{2} BD = \frac{\sqrt{2}}{2}$, so

$$\cos\theta = \frac{\frac{\sqrt{2}}{2}}{1} = \boldsymbol{\frac{\sqrt{2}}{2}}\ .$$

〈Note〉 i.e.,　$\angle ABO = 45°$.

２０

In a regular quadrilateral pyramid A-BCDE with length 1 on one side,
find the value of $\cos\theta$ if the angle between the planes ABC and BCDE is θ.



solution

The intersection of planes ABC and BCDE is straight line BC.

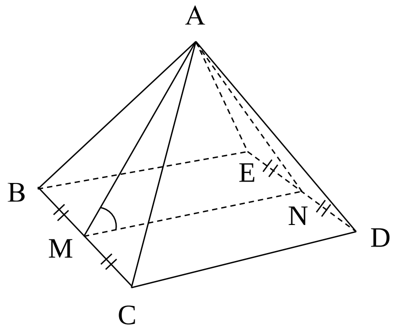
Let M be the midpoint of side BC.

△ABC can be thought of as an isosceles triangle with AB＝AC.

Therefore, AM⊥BC.

Let N be the midpoint of side ED, BC⊥MN.

Thus, $\theta=\angle\mathrm{AMN}$.



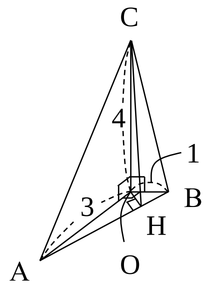
Here, $\mathrm{AM}=\mathrm{AN}=\dfrac{\sqrt{3}}{2}$, MN＝1, so

$$\cos\theta=\frac{\mathrm{MA}^2+\mathrm{MN}^2-\mathrm{AN}^2}{2\mathrm{MA}\cdot\mathrm{MN}}=\frac{\left(\dfrac{\sqrt{3}}{2}\right)^2+1^2-\left(\dfrac{\sqrt{3}}{2}\right)^2}{2\cdot\dfrac{\sqrt{3}}{2}\cdot1}=\frac{1}{\sqrt{3}}=\frac{\sqrt{\mathbf{3}}}{\mathbf{3}}.$$

21

There are line segments OA, OB, and OC perpendicular to each other,
OA = 3, OB = 1, and OC = 4.
When drawing a perpendicular line OH from point O to line segment AB,
answer the following questions.

(1) Find the length of the line segment OH.
(2) Find the length of the line segment CH.
(3) Find the area of △ABC.



solution

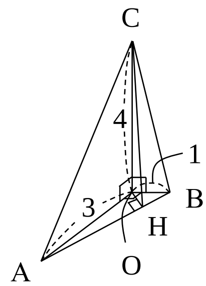
(1) Focusing on the area of △OAB,

$$\frac{1}{2}\times OA\times OB=\frac{1}{2}\times AB\times OH \quad \cdots\cdots① .$$

Since OA = 3, OB = 1, and $AB=\sqrt{3^2+1^2}=\sqrt{10}$,

$$\frac{1}{2}\times 3\times 1=\frac{1}{2}\times\sqrt{10}\times OH \quad \text{from ①.}$$

Therefore, $OH=\frac{3}{\sqrt{10}}=\mathbf{\frac{3\sqrt{10}}{10}}$.



(2) △OCH is a right triangle with angle COH = 90° , so

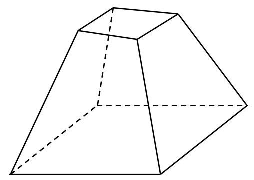
$$CH=\sqrt{OC^2+OH^2}=\sqrt{4^2+\left(\frac{3}{\sqrt{10}}\right)^2}=\frac{13}{\sqrt{10}}=\mathbf{\frac{13\sqrt{10}}{10}}.$$

(3) Since OC⊥ plane OAB and OH⊥AB, by the trilinear theorem, CH⊥AB.

Thus, the area of △ABC is $\frac{1}{2}\times AB\times CH=\frac{1}{2}\times\sqrt{10}\times\frac{13}{\sqrt{10}}=\mathbf{\frac{13}{2}}$.

22

Verify that Euler's polyhedron theorem holds for the polyhedron shown in the figure on the right.



solution

The number of vertices v is 8,

the number of edges e is 12,

and the number of faces f is 6, so

$$v-e+f=8-12+6=2\ .$$

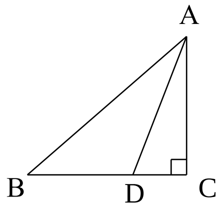
Therefore, **Euler's polyhedron theorem holds.**

Study

Answer the following questions.

(1) Right triangle ABC with angle C = 90° ,
let D be a point on side BC, prove that

AB＞AD .

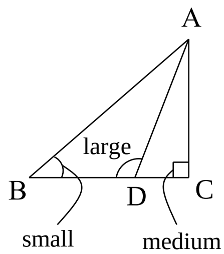


(2) Find if there exists a triangle whose 3 sides have the following lengths.

① 3, 5, 7 ② 1, 2, 3

solution

(1) In △ABC, ∠C＝90° , so ∠B＜∠C ……① .
Since ∠ADB is the outside angle of △ADC, ∠ADB＝∠ACD＋∠CAD .
Therefore, ∠ADB＞∠C ……② .
From ① and ②, ∠ADB＞∠B .
Thus, AB > AD due to the side and angle of the triangle.



(2) ① Since 3＋5＞7, 5＋7＞3, and 7＋3＞5,
triangles of length 3, 5, and 7 **exist**.

② Since 3＝1＋2, **there are no triangles** with 3 sides of length 1, 2, and 3.