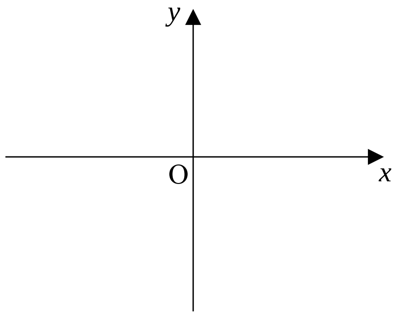
Trigonometric function

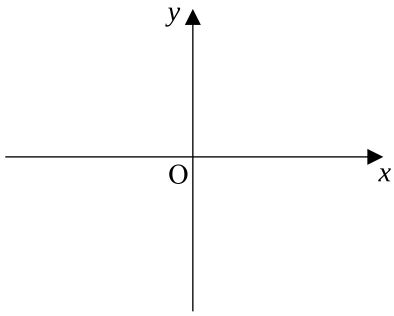
1

In the coordinate plane with point O as the origin, take the positive part of the x-axis as the starting line and illustrate the radius of motion OP rotated by the following angle. Also, express the general angle θ represented by the radius of motion OP in the form $\theta = \alpha + 360° \times n$ ($0° \leqq \alpha < 360°$, n is an integer), and answer in what quadrant the angle is.

(1) 800°

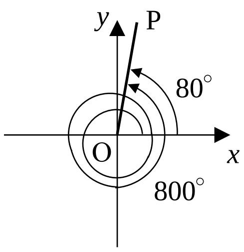


(2) −200°



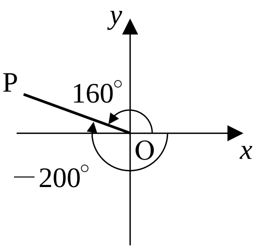
solution

(1)



800° ＝80° ＋360° ×2, angle in the first quadrant

(2)



−200° ＝160° ＋360° ×(−1), angle in the second quadrant

2 Rewrite the following angles in degrees to arc degrees and arc degrees to degrees, respectively.

(1) 135° (2) -108° (3) $\frac{\pi}{2}$ (4) $-\frac{13}{10}\pi$

solution

(1) $135^\circ = 135 \times \frac{\pi}{180} = \boldsymbol{\frac{3}{4}\pi}$

(2) $-108^\circ = -108 \times \frac{\pi}{180} = \boldsymbol{-\frac{3}{5}\pi}$

(3) $\frac{\pi}{2} = \frac{1}{2} \times 180^\circ = \mathbf{90}^\circ$

(4) $-\frac{13}{10}\pi = -\frac{13}{10} \times 180^\circ = \mathbf{-234}^\circ$

3

Find the arc length l and area S of a fan shape whose radius is 9 and whose central angle is $\frac{2}{3}\pi$.

solution

$\boldsymbol{l} = 9 \cdot \frac{2}{3}\pi = \boldsymbol{6\pi}$

$\boldsymbol{S} = \frac{1}{2} \cdot 9^2 \cdot \frac{2}{3}\pi = \boldsymbol{27\pi}$

Alternative solution for S

$S = \frac{1}{2} \cdot 9 \cdot 6\pi = 27\pi$

Let radius r and central angle θ be the length l of the arc of the fan shape and the area S of the fan shape.

$\boldsymbol{l = r\theta}$

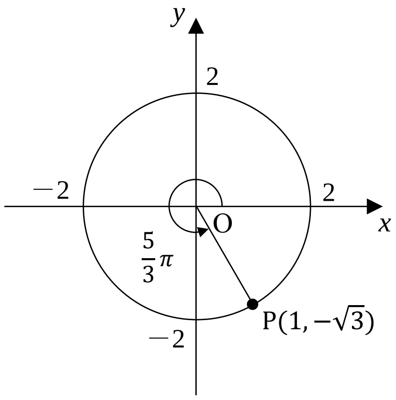
$\boldsymbol{S = \frac{1}{2}r^2\theta = \frac{1}{2}rl}$

4 Find the values of $\sin\theta$, $\cos\theta$, and $\tan\theta$, respectively, when θ has the following values.

(1) $\frac{5}{3}\pi$ (2) $-\frac{3}{4}\pi$

solution

(1) Let P be the intersection of the radius of motion of $\frac{5}{3}\pi$ and the circle of radius 2 centered at the origin,



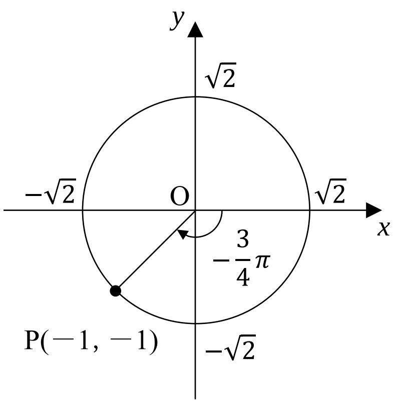
$P(1,\ -\sqrt{3})$, so we have

$$\sin\frac{5}{3}\pi = \frac{-\sqrt{3}}{2} = -\frac{\sqrt{3}}{2},$$

$$\cos\frac{5}{3}\pi = \frac{1}{2},$$

$$\tan\frac{5}{3}\pi = \frac{-\sqrt{3}}{1} = -\sqrt{3}.$$

(2) Let P be the intersection of the radius of motion of $-\frac{3}{4}\pi$ and the circle of radius $\sqrt{2}$ centered at the origin,



$P(-1,\ -1)$, so we have

$$\sin\left(-\frac{3}{4}\pi\right) = \frac{-1}{\sqrt{2}} = -\frac{1}{\sqrt{2}},$$

$$\cos\left(-\frac{3}{4}\pi\right) = \frac{-1}{\sqrt{2}} = -\frac{1}{\sqrt{2}},$$

$$\tan\left(-\frac{3}{4}\pi\right) = \frac{-1}{-1} = 1.$$

5

If θ is an angle in the fourth quadrant and $\cos\theta = \frac{1}{3}$, find the values of $\sin\theta$ and $\tan\theta$, respectively.

solution

From $\sin^2\theta + \cos^2\theta = 1$, $\sin^2\theta = 1 - \cos^2\theta = 1 - \left(\frac{1}{3}\right)^2 = \frac{8}{9}$.

Since θ is the angle in the fourth quadrant, $\sin\theta < 0$.

Therefore, $\boldsymbol{\sin\theta} = -\sqrt{\frac{8}{9}} = -\boldsymbol{\frac{2\sqrt{2}}{3}}$.

Also, $\boldsymbol{\tan\theta} = \frac{\sin\theta}{\cos\theta} = \left(-\frac{2\sqrt{2}}{3}\right) \div \frac{1}{3} = \boldsymbol{-2\sqrt{2}}$.

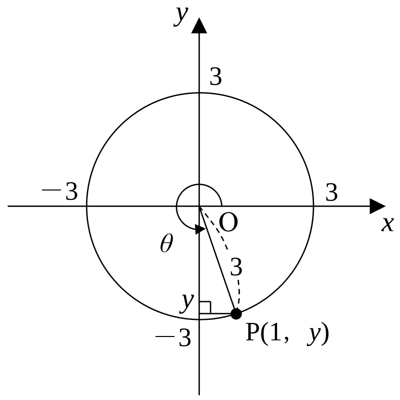
Alternative solution Draw a diagram to find it.

From the conditions, we take the point P(1, y) in the fourth quadrant where $r = 3$ and $x = 1$.

At this time, $y = -\sqrt{3^2 - 1^2} = -2\sqrt{2}$.

Therefore, $\sin\theta = -\frac{2\sqrt{2}}{3}$,

$\tan\theta = \frac{-2\sqrt{2}}{1} = -2\sqrt{2}$.



6

When $\sin\theta+\cos\theta=\frac{1}{2}$, find the value of the following expression.

(1) $\sin\theta\cos\theta$ (2) $\sin^3\theta+\cos^3\theta$

solution

(1) Squaring both sides of $\sin\theta+\cos\theta=\frac{1}{2}$ yields $\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta=\frac{1}{4}$.

From $\sin^2\theta+\cos^2\theta=1$, we get $1+2\sin\theta\cos\theta=\frac{1}{4}$. Therefore, $\sin\theta\cos\theta=-\boldsymbol{\frac{3}{8}}$.

(2) $\sin^3\theta+\cos^3\theta=(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)=\frac{1}{2}\left\{1-\left(-\frac{3}{8}\right)\right\}=\boldsymbol{\frac{11}{16}}$

Alternative solution

$\sin^3\theta+\cos^3\theta=(\sin\theta+\cos\theta)^3-3\sin\theta\cos\theta\,(\sin\theta+\cos\theta)$

$$=\left(\frac{1}{2}\right)^3-3\cdot\left(-\frac{3}{8}\right)\cdot\frac{1}{2}=\frac{1}{8}+\frac{9}{16}=\frac{11}{16}$$

7 Find the following values.

(1) $\sin\frac{100}{3}\pi$ (2) $\tan\left(-\frac{3}{4}\pi\right)$ (3) $\sin\frac{3}{10}\pi + \cos\frac{4}{5}\pi$

solution

(1) $\sin\frac{100}{3}\pi = \sin\left(\frac{4}{3}\pi + 32\pi\right) = \sin\frac{4}{3}\pi = -\boldsymbol{\frac{\sqrt{3}}{2}}$

(2) $\tan\left(-\frac{3}{4}\pi\right) = -\tan\frac{3}{4}\pi = -(-1) = \mathbf{1}$

(3) $\sin\frac{3}{10}\pi + \cos\frac{4}{5}\pi = \sin\left(\frac{\pi}{2} - \frac{\pi}{5}\right) + \cos\left(\pi - \frac{\pi}{5}\right) = \cos\frac{\pi}{5} - \cos\frac{\pi}{5} = \mathbf{0}$

8

(1) Graph the following functions. Find its period.

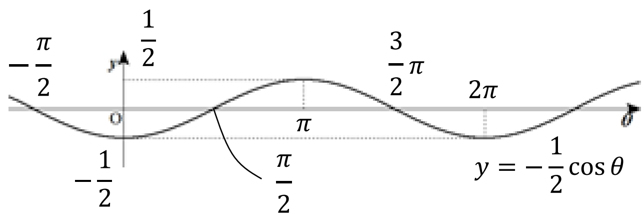
① $y=-\frac{1}{2}\cos\theta$ ② $y=\tan 2\theta$ ③ $y=\sin\left(\theta+\frac{\pi}{2}\right)+1$

(2) For the functions ① through ③ in (1), answer which are even functions and which are odd functions, respectively.

solution

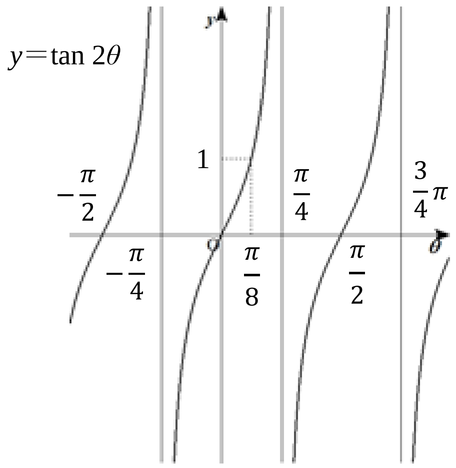
(1) ① The graph is shown on the right.

The period is $\boldsymbol{2\pi}$.



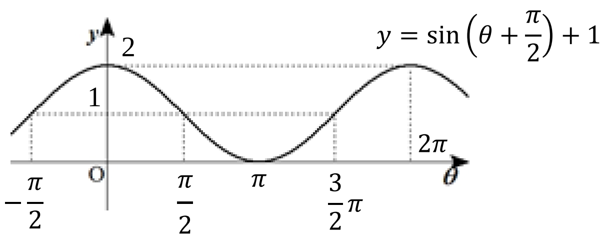
② The graph is shown on the right.

The period is $\boldsymbol{\frac{\pi}{2}}$.



③ The graph is shown on the right.

The period is $\boldsymbol{2\pi}$.



(2) In ① through ③, let $y=f(\theta)$.

① $f(-\theta)=-\frac{1}{2}\cos(-\theta)=-\frac{1}{2}\cos\theta=f(\theta)$

② $f(-\theta)=\tan(-2\theta)=-\tan 2\theta=-f(\theta)$

③ $f(-\theta)=\sin\left(-\theta+\frac{\pi}{2}\right)+1=\cos\theta+1=\sin\left(\theta+\frac{\pi}{2}\right)+1=f(\theta)$

Therefore, **the even functions are ① and ③, and the odd functions are ② .**

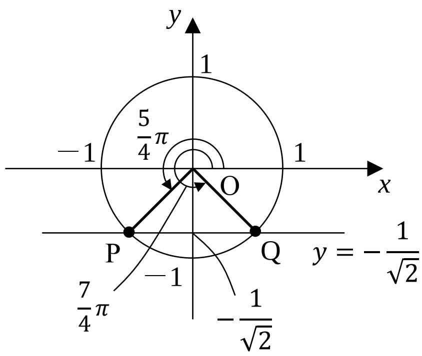
9 Solve the following equations and inequalities for $0 \leqq \theta < 2\pi$.

(1) $\sin\theta = -\dfrac{1}{\sqrt{2}}$ (2) $\cos\theta > \dfrac{1}{2}$

solution

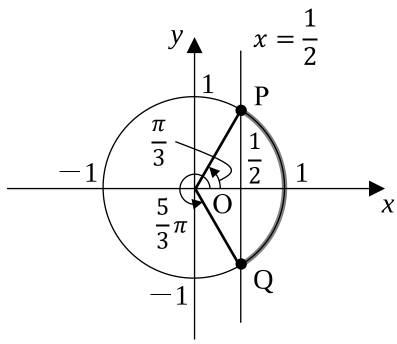
(1) Let the intersection of the line $y = -\dfrac{1}{\sqrt{2}}$ and the unit circle P and Q as shown in the figure on the right . Therefore, the required θ is

$$\boldsymbol{\theta = \frac{5}{4}\pi,\ \ \frac{7}{4}\pi}\ .$$



(2) Let the intersection of the line $x = \dfrac{1}{2}$ and the unit circle P and Q as shown in the figure on the right . Therefore, the range of θ satisfying the inequality is

$$\boldsymbol{0 \leqq \theta < \frac{\pi}{3},\ \ \frac{5}{3}\pi < \theta < 2\pi}\ .$$



１０

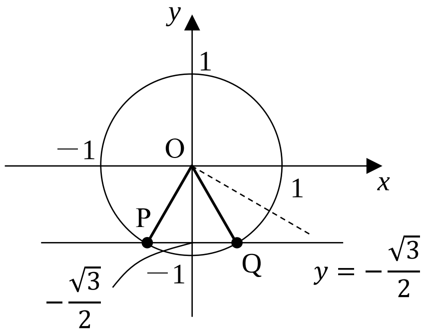
(1) Solve the equation $2\sin\left(\theta-\frac{\pi}{6}\right)=-\sqrt{3}$ for $0 \leqq \theta < 2\pi$.

(2) Solve the following equations and inequalities for $0 \leqq \theta < 2\pi$.

① $2\sin^2\theta+3\cos\theta-3=0$ ② $2\sin^2\theta+3\cos\theta-3\geqq 0$

solution

(1) If $X=\theta-\frac{\pi}{6}$, then $\sin X=-\frac{\sqrt{3}}{2}$.



Here , the possible values of X range from

$0 \leqq \theta < 2\pi$ to $-\frac{\pi}{6} \leqq \theta-\frac{\pi}{6} < 2\pi-\frac{\pi}{6}$,

thus $-\frac{\pi}{6} \leqq X < \frac{11}{6}\pi$.

From this, the X we seek is $X=\frac{4}{3}\pi,\ \frac{5}{3}\pi$.

That is , $\theta-\frac{\pi}{6}=\frac{4}{3}\pi,\ \frac{5}{3}\pi$. From the above, $\boldsymbol{\theta=\frac{3}{2}\pi,\ \frac{11}{6}\pi}$.

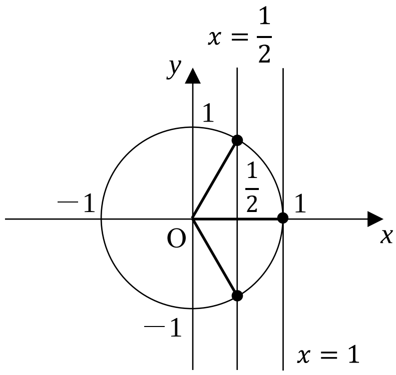
(2) ① $\sin^2\theta+\cos^2\theta=1$ to $\sin^2\theta=1-\cos^2\theta$.

Thus, the given equation is $2(1-\cos^2\theta)+3\cos\theta-3=0$,

$2-2\cos^2\theta+3\cos\theta-3=0$,

$2\cos^2\theta-3\cos\theta+1=0$,

$(2\cos\theta-1)(\cos\theta-1)=0$.



From this, $\cos\theta=\frac{1}{2},\ 1$.

From $0 \leqq \theta < 2\pi$, $\boldsymbol{\theta=0,\ \frac{\pi}{3},\ \frac{5}{3}\pi}$.

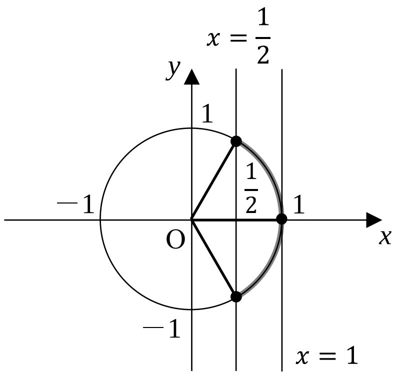
② Transforming as in ①, we obtain

$2(1-\cos^2\theta)+3\cos\theta-3\geqq 0$,

$2-2\cos^2\theta+3\cos\theta-3\geqq 0$,

$2\cos^2\theta-3\cos\theta+1\leqq 0$,

$(2\cos\theta-1)(\cos\theta-1)\leqq 0$.



From this, $\frac{1}{2} \leqq \cos\theta \leqq 1$.

Since $0 \leqq \theta < 2\pi$, the range of θ to be sought is

$\boldsymbol{0 \leqq \theta \leqq \frac{\pi}{3},\ \frac{5}{3}\pi \leqq \theta < 2\pi}$.

1 1

Find the maximum and minimum values of the function $y=\sin^2\theta+\cos\theta$ when $0\leqq\theta<2\pi$.

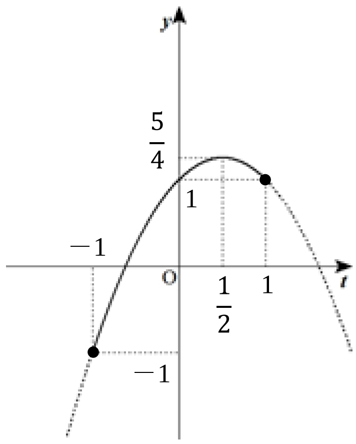
Also, find the value of θ at that time.

solution

$\sin^2\theta+\cos^2\theta=1$ to $\sin^2\theta=1-\cos^2\theta$.

Thus, the given function can be transformed to

$$y=(1-\cos^2\theta)+\cos\theta$$
$$=-\cos^2\theta+\cos\theta+1.$$



Where $\cos\theta=t$ and $-1\leqq t\leqq 1$.

The given function is $y=-t^2+t+1$

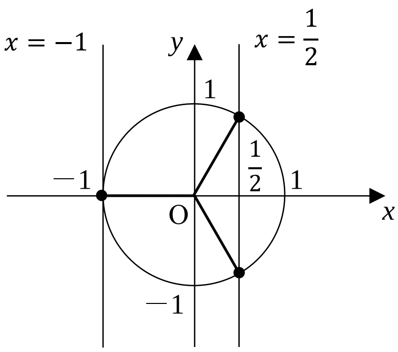
$$=-(t^2-t)+1$$
$$=-\left\{\left(t-\frac{1}{2}\right)^2-\frac{1}{4}\right\}+1$$
$$=-\left(t-\frac{1}{2}\right)^2+\frac{1}{4}+1$$
$$=-\left(t-\frac{1}{2}\right)^2+\frac{5}{4}.$$

Thus, y has a maximum value of $\frac{5}{4}$ when $t=\frac{1}{2}$ and a minimum value of -1 when $t=-1$.

Where $0\leqq\theta<2\pi$,

so that $t=\frac{1}{2}$, since $\cos\theta=\frac{1}{2}$ to $\theta=\frac{\pi}{3},\ \frac{5}{3}\pi$,

so that $t=-1$, since $\cos\theta=-1$ to $\theta=\pi$.



From the above, **the maximum value is $\frac{5}{4}$ when $\theta=\frac{\pi}{3},\ \frac{5}{3}\pi$,**

and the minimum value is -1 when $\theta=\pi$.

12 Find the following values.

(1) $\sin 15°$ (2) $\cos 195°$ (3) $\tan\frac{5}{12}\pi$

solution

(1) $\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \boldsymbol{\frac{\sqrt{6}-\sqrt{2}}{4}}$$

Alternative solution

$15° = 60° - 45°$ may be considered.

$$\sin 15° = \sin(60° - 45°) = \sin 60° \cos 45° - \cos 60° \sin 45°$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} - \frac{1}{2}\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

(2) $\cos 195° = \cos(45° + 150°) = \cos 45° \cos 150° - \sin 45° \sin 150°$

$$= \frac{1}{\sqrt{2}}\cdot\left(-\frac{\sqrt{3}}{2}\right) - \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{-\sqrt{3}-1}{2\sqrt{2}} = \boldsymbol{-\frac{\sqrt{6}+\sqrt{2}}{4}}$$

(3) $\frac{5}{12}\pi = \frac{\pi}{6} + \frac{\pi}{4}$, and therefore

$\frac{5}{12}\pi$ in the arc degree method is

$\frac{5}{12} \times 180° = 75°$ in the degree method.

$75° = 30° + 45°$ can be considered.

$$\tan\frac{5}{12}\pi = \tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$$

$$= \frac{\tan\frac{\pi}{6} + \tan\frac{\pi}{4}}{1 - \tan\frac{\pi}{6}\cdot\tan\frac{\pi}{4}}$$

$$= \frac{\frac{1}{\sqrt{3}} + 1}{1 - \frac{1}{\sqrt{3}}\cdot 1} = \frac{1+\sqrt{3}}{\sqrt{3}-1} = \frac{(1+\sqrt{3})^2}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{1+2\sqrt{3}+3}{3-1} = \frac{4+2\sqrt{3}}{2} = \boldsymbol{2+\sqrt{3}}.$$

13

$0<\alpha<\frac{\pi}{2}$, $\pi<\beta<\frac{3}{2}\pi$ and $\cos\alpha=\frac{12}{13}$, $\sin\beta=-\frac{3}{5}$, find the following values.

(1) $\sin(\alpha-\beta)$ (2) $\cos(\alpha-\beta)$

solution

(1) From $0<\alpha<\frac{\pi}{2}, \sin\alpha>0$. Therefore, $\sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{1-\left(\frac{12}{13}\right)^2}=\sqrt{\frac{169-144}{13^2}}=\frac{5}{13}$.

From $\pi<\beta<\frac{3}{2}\pi, \cos\beta<0$. Therefore, $\cos\beta=-\sqrt{1-\sin^2\beta}=-\sqrt{1-\left(-\frac{3}{5}\right)^2}=-\frac{4}{5}$.

Thus, $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta=\frac{5}{13}\cdot\left(-\frac{4}{5}\right)-\frac{12}{13}\cdot\left(-\frac{3}{5}\right)=\mathbf{\frac{16}{65}}$.

(2) From (1), $\sin\alpha=\frac{5}{13}$, $\cos\beta=-\frac{4}{5}$.

Therefore, $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta=\frac{12}{13}\cdot\left(-\frac{4}{5}\right)+\frac{5}{13}\cdot\left(-\frac{3}{5}\right)=-\mathbf{\frac{63}{65}}$.

14 Find the acute angle θ formed by the two lines $y=5x$ and $2x=3y$.

solution

The slope of the line $y=5x$ is 5 .

Line $2x = 3y$ can be transformed into line $y = \frac{2}{3}x$, so the slope is $\frac{2}{3}$.

Let α and β be the angles between the two lines and the positive part of the $x-$axis, respectively .

$\tan\alpha = 5$ and $\tan\beta = \frac{2}{3}$.

Therefore, $\tan\theta = \tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \dfrac{5 - \frac{2}{3}}{1 + 5\cdot\frac{2}{3}} = 1$.

From $0 < \theta < \frac{\pi}{2}$, $\boldsymbol{\theta = \frac{\pi}{4}}$.

15

Find the values of $\sin 2\alpha$, $\cos 2\alpha$, and $\tan 2\alpha$ when $\frac{\pi}{2} < \alpha < \pi$ and $\sin\alpha = \frac{1}{4}$.

solution

From $\frac{\pi}{2} < \alpha < \pi$, $\cos\alpha < 0$. Therefore, $\cos\alpha = -\sqrt{1-\sin^2\alpha} = -\sqrt{1-\left(\frac{1}{4}\right)^2} = -\frac{\sqrt{15}}{4}$.

Thus, $\mathbf{\sin 2\alpha} = 2\sin\alpha\cos\alpha = 2\cdot\frac{1}{4}\cdot\left(-\frac{\sqrt{15}}{4}\right) = -\frac{\sqrt{\mathbf{15}}}{\mathbf{8}}$,

$\mathbf{\cos 2\alpha} = 1 - 2\sin^2\alpha = 1 - 2\cdot\left(\frac{1}{4}\right)^2 = \frac{\mathbf{7}}{\mathbf{8}}$,

$\mathbf{\tan 2\alpha} = \frac{\sin 2\alpha}{\cos 2\alpha} = \left(-\frac{\sqrt{15}}{8}\right) \div \frac{7}{8} = -\frac{\sqrt{\mathbf{15}}}{\mathbf{7}}$.

１６

Find the values of $\sin\frac{\alpha}{2}, \cos\frac{\alpha}{2}$, and $\tan\frac{\alpha}{2}$ when $\frac{3}{2}\pi < \alpha < 2\pi$ and $\sin\alpha = -\frac{4}{5}$.

solution

From $\frac{3}{2}\pi < \alpha < 2\pi, \cos\alpha > 0$. Therefore, $\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(-\frac{4}{5}\right)^2} = \frac{3}{5}$.

Moreover, from $\frac{3}{4}\pi < \frac{\alpha}{2} < \pi, \sin\frac{\alpha}{2} > 0, \cos\frac{\alpha}{2} < 0, \tan\frac{\alpha}{2} < 0$.

Thus, $\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2} = \frac{1-\frac{3}{5}}{2} = \frac{1}{5}$.

$\sin\frac{\alpha}{2} > 0$, so $\boldsymbol{\sin\frac{\alpha}{2} = \sqrt{\frac{1}{5}} = \frac{\sqrt{5}}{5}}$.

$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2} = \frac{1+\frac{3}{5}}{2} = \frac{4}{5}$.

$\cos\frac{\alpha}{2} < 0$, so $\boldsymbol{\cos\frac{\alpha}{2} = -\sqrt{\frac{4}{5}} = -\frac{2\sqrt{5}}{5}}$.

$\boldsymbol{\tan\frac{\alpha}{2}} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \frac{\sqrt{5}}{5} \div \left(-\frac{2\sqrt{5}}{5}\right) = \boldsymbol{-\frac{1}{2}}$.

17 Solve the following equations and inequalities for $0 \leqq \theta < 2\pi$.

(1) $\sin 2\theta = -\sqrt{2}\cos\theta$ (2) $\cos 2\theta < 3\cos\theta + 1$

solution

(1) From $\sin 2\theta = 2\sin\theta\cos\theta$, the given equation is $2\sin\theta\cos\theta = -\sqrt{2}\cos\theta$.

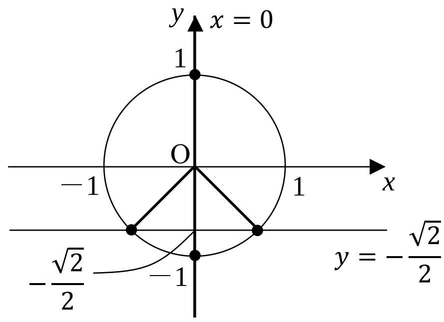
To summarize, we have $\cos\theta\left(2\sin\theta + \sqrt{2}\right) = 0$.

Therefore, $\cos\theta = 0$ or $\sin\theta = -\dfrac{\sqrt{2}}{2}$.

Since $0 \leqq \theta < 2\pi$,

solving for $\cos\theta = 0$ yields $\theta = \dfrac{\pi}{2},\ \dfrac{3}{2}\pi$.

Solving for $\sin\theta = -\dfrac{\sqrt{2}}{2}$ yields $\theta = \dfrac{5}{4}\pi,\ \dfrac{7}{4}\pi$.



From the above, we obtain $\boldsymbol{\theta = \dfrac{\pi}{2},\ \dfrac{5}{4}\pi,\ \dfrac{3}{2}\pi,\ \dfrac{7}{4}\pi}$.

(2) From $\cos 2\theta = 2\cos^2\theta - 1$, transforming the given equation, $2\cos^2\theta - 1 < 3\cos\theta + 1$,

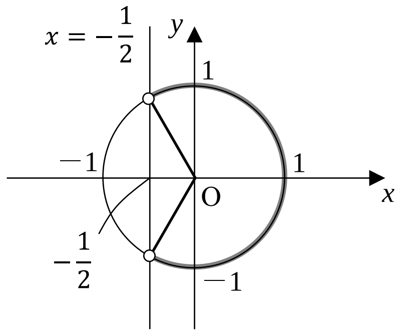
$2\cos^2\theta - 3\cos\theta - 2 < 0$,

$(\cos\theta - 2)(2\cos\theta + 1) < 0$.

Since $-1 \leqq \cos\theta \leqq 1$, $\cos\theta - 2 < 0$ at all times .

Therefore, $2\cos\theta + 1 > 0$. That is, $\cos\theta > -\dfrac{1}{2}$.

Thus, $\boldsymbol{0 \leqq \theta < \dfrac{2}{3}\pi,\ \dfrac{4}{3}\pi < \theta < 2\pi}$.



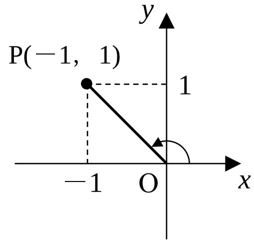
18 Transform the following equation into the form $r\sin(\theta+\alpha)$. However, $r>0$ and $-\pi<\alpha\leqq\pi$.

(1) $-\sin\theta+\cos\theta$ (2) $\sqrt{3}\sin\theta-3\cos\theta$

solution

(1) $-\sin\theta+\cos\theta$, take the point $\mathrm{P}(-1,\ 1)$ as shown in the figure on the right, which is

$$r=\sqrt{(-1)^2+1^2}=\sqrt{2},\quad \alpha=\frac{3}{4}\pi .$$



Therefore, $-\sin\theta+\cos\theta=\boldsymbol{\sqrt{2}\sin\left(\theta+\frac{3}{4}\pi\right)}$.

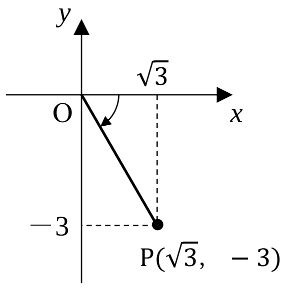
〈Note〉 By using the additive theorem on the right-hand side, we can check if we have deformed it correctly.

$$\sqrt{2}\sin\left(\theta+\frac{3}{4}\pi\right)=\sqrt{2}\left(\sin\theta\cos\frac{3}{4}\pi+\cos\theta\sin\frac{3}{4}\pi\right)=\sqrt{2}\left\{\sin\theta\cdot\left(-\frac{1}{\sqrt{2}}\right)+\cos\theta\cdot\frac{1}{\sqrt{2}}\right\}=-\sin\theta+\cos\theta$$

(2) $\sqrt{3}\sin\theta-3\cos\theta$, take the point $\mathrm{P}(\sqrt{3},\ -3)$ as shown in the figure on the right, which is

$$r=\sqrt{\left(\sqrt{3}\right)^2+(-3)^2}=2\sqrt{3},$$

$$\alpha=-\frac{\pi}{3}.$$



Therefore, $\sqrt{3}\sin\theta-3\cos\theta=\boldsymbol{2\sqrt{3}\sin\left(\theta-\frac{\pi}{3}\right)}$.

19 Solve the following equations and inequalities for $0 \leqq \theta < 2\pi$.

(1) $\sin\theta - \sqrt{3}\cos\theta - 1 = 0$ (2) $\sqrt{2}\sin\theta + \sqrt{2}\cos\theta \leqq -\sqrt{3}$

solution

(1) From the figure on the right,

$$\sin\theta - \sqrt{3}\cos\theta = 2\sin\left(\theta - \frac{\pi}{3}\right).$$

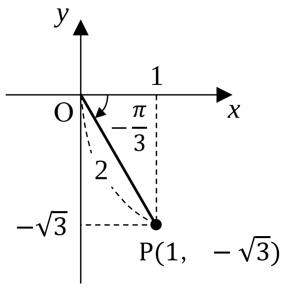
Therefore, transforming the given equation, we obtain

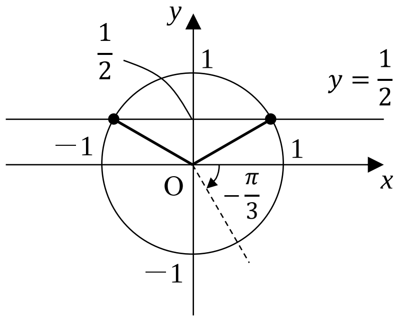
$$\sin\left(\theta - \frac{\pi}{3}\right) = \frac{1}{2}.$$

From $-\frac{\pi}{3} \leqq \theta - \frac{\pi}{3} < \frac{5}{3}\pi$,

$$\theta - \frac{\pi}{3} = \frac{\pi}{6},\ \frac{5}{6}\pi .$$

Thus, $\boldsymbol{\theta = \frac{\pi}{2},\ \frac{7}{6}\pi}$.





(2) From the figure on the right,

$$\sqrt{2}\sin\theta + \sqrt{2}\cos\theta = 2\sin\left(\theta + \frac{\pi}{4}\right).$$

Therefore, transforming the given equation, we obtain

$$\sin\left(\theta + \frac{\pi}{4}\right) \leqq -\frac{\sqrt{3}}{2}.$$

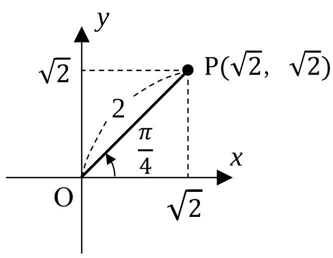
From $\frac{\pi}{4} \leqq \theta + \frac{\pi}{4} < \frac{9}{4}\pi$, solving for

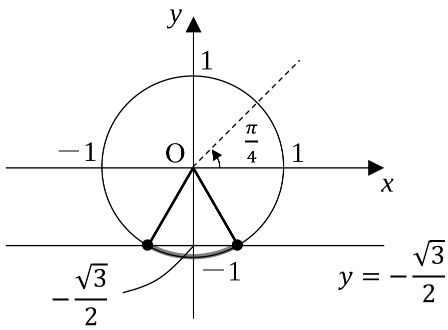
$$\sin\left(\theta + \frac{\pi}{4}\right) = -\frac{\sqrt{3}}{2},$$

we get $\theta + \frac{\pi}{4} = \frac{4}{3}\pi,\ \frac{5}{3}\pi$.

From this, we get $\frac{4}{3}\pi \leqq \theta + \frac{\pi}{4} \leqq \frac{5}{3}\pi$.

Thus, $\boldsymbol{\frac{13}{12}\pi \leqq \theta \leqq \frac{17}{12}\pi}$.



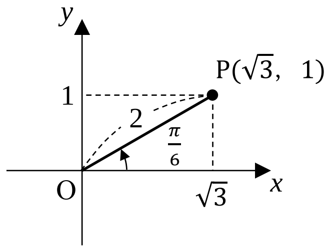


２０

Find the maximum and minimum values of the function $y = \sqrt{3}\sin\theta + \cos\theta - 1$ when $0 \leqq \theta < 2\pi$.
Also, find the value of θ at that time.

solution

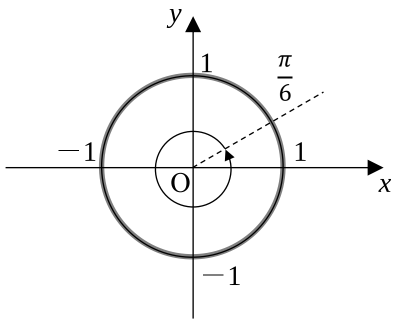
From the figure on the right,



$$\sqrt{3}\sin\theta + \cos\theta = 2\sin\left(\theta + \frac{\pi}{6}\right).$$

Therefore, the given function is $y = 2\sin\left(\theta + \frac{\pi}{6}\right) - 1$.

$0 \leqq \theta < 2\pi$, so $\frac{\pi}{6} \leqq \theta + \frac{\pi}{6} < \frac{13}{6}\pi$.



$-1 \leqq \sin\left(\theta + \frac{\pi}{6}\right) \leqq 1$, so $-3 \leqq y \leqq 1$.

$\sin\left(\theta + \frac{\pi}{6}\right) = 1$, then $\theta + \frac{\pi}{6} = \frac{\pi}{2}$, therefore $\theta = \frac{\pi}{3}$.

$\sin\left(\theta + \frac{\pi}{6}\right) = -1$, then $\theta + \frac{\pi}{6} = \frac{3}{2}\pi$, therefore $\theta = \frac{4}{3}\pi$.

Thus, **it takes the maximum value 1 when $\boldsymbol{\theta = \frac{\pi}{3}}$**

and the minimum value $\boldsymbol{-3}$ when $\boldsymbol{\theta = \frac{4}{3}\pi}$.

Study 1

If the equation $\sin^2\theta + \cos\theta - a = 0$ has three solutions with $0 \leqq \theta < 2\pi$, find the value of the constant a.

solution

$\sin^2\theta + \cos^2\theta = 1$, so $\sin^2\theta = 1 - \cos^2\theta$.

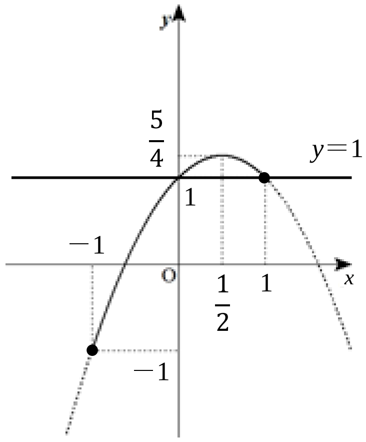
If $\cos\theta = x$, then $-1 \leqq x \leqq 1$.

The given equation is $1 - x^2 + x = a$.

Let $f(x) = 1 - x^2 + x$ be

$$f(x) = -x^2 + x + 1 = -(x^2 - x) + 1$$

$$= -\left\{\left(x - \frac{1}{2}\right)^2 - \frac{1}{4}\right\} + 1 = -\left(x - \frac{1}{2}\right)^2 + \frac{5}{4}.$$



The graph of the function $y = f(x)$ is shown on the right .

The equation for a given θ has three solutions

when the graph of the function $y = f(x)$ and the line $y = a$ intersect

at $x = 1$ and $-1 < x < 1$.

Or when the curve and the line intersect

at $x = -1$ and $-1 < x < 1$.

From the graph on the right, $y = f(x)$ and $y = 1$ intersect

at $x = 0$ and 1, so the value of a to be found is $\boldsymbol{a = 1}$.

When $x = 0$ and $1, \theta = \frac{\pi}{2}, \ \frac{3}{2}\pi, \ 0,$ satisfying the subject of having three solutions .

Study 2 Find the following values.

(1) $\sin 105° \ \cos 15°$ (2) $\cos 15° \ \cos 75°$

(3) $\sin 15° + \sin 75°$ (4) $\cos 15° - \cos 105°$

solution

(1) $\sin 105° \ \cos 15° = \frac{1}{2}\{\sin(105° + 15°) + \sin(105° - 15°)\} = \frac{1}{2}(\sin 120° + \sin 90°)$

$$= \frac{1}{2}\left(\frac{\sqrt{3}}{2} + 1\right) = \mathbf{\frac{\sqrt{3}+2}{4}}$$

(2) $\cos 15° \ \cos 75° = \frac{1}{2}\{\cos(15° + 75°) + \cos(15° - 75°)\} = \frac{1}{2}\{\cos 90° + \cos(-60°)\}$

$$= \frac{1}{2}\left(0 + \frac{1}{2}\right) = \mathbf{\frac{1}{4}}$$

(3) $\sin 15° + \sin 75° = 2\sin\frac{15° + 75°}{2}\cos\frac{15° - 75°}{2} = 2\sin 45° \ \cos(-30°) = 2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} = \mathbf{\frac{\sqrt{6}}{2}}$

(4) $\cos 15° - \cos 105° = -2\sin\frac{15° + 105°}{2}\sin\frac{15° - 105°}{2}$

$$= -2\sin 60° \ \sin(-45°) = -2 \cdot \frac{\sqrt{3}}{2} \cdot \left(-\frac{1}{\sqrt{2}}\right) = \mathbf{\frac{\sqrt{6}}{2}}$$

Study 3 When $0 \leqq \theta < 2\pi$, answer the following questions.

(1) Find the maximum and minimum values of the function $y = \sin\theta\cos\theta - \sqrt{3}\sin^2\theta$.

Also, find the value of θ at that time.

(2) Find the maximum and minimum values of the function $y = \sin 2\theta - 2\sin\theta + 2\cos\theta$.

Also, find the value of θ at that time.

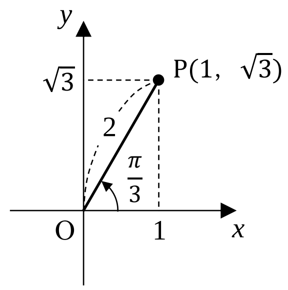
solution

(1) $\sin\theta\cos\theta = \dfrac{\sin 2\theta}{2}$, $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$, so

$$y = \sin\theta\cos\theta - \sqrt{3}\sin^2\theta = \frac{\sin 2\theta}{2} - \sqrt{3}\cdot\frac{1-\cos 2\theta}{2}$$

$$= \frac{1}{2}\left(\sin 2\theta + \sqrt{3}\cos 2\theta\right) - \frac{\sqrt{3}}{2}$$

$$= \frac{1}{2}\cdot 2\sin\left(2\theta + \frac{\pi}{3}\right) - \frac{\sqrt{3}}{2} = \sin\left(2\theta + \frac{\pi}{3}\right) - \frac{\sqrt{3}}{2}.$$



$0 \leqq \theta < 2\pi$, then $\dfrac{\pi}{3} \leqq 2\theta + \dfrac{\pi}{3} < \dfrac{13}{3}\pi$.

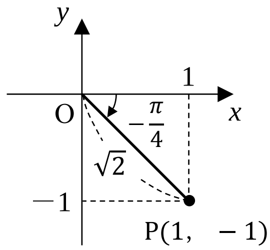
Therefore, $2\theta + \dfrac{\pi}{3} = \dfrac{\pi}{2}$, $\dfrac{5}{2}\pi$, that is, **y takes the maximum value $1 - \dfrac{\sqrt{3}}{2}$ when $\theta = \dfrac{\pi}{12}$, $\dfrac{13}{12}\pi$,**

$2\theta + \dfrac{\pi}{3} = \dfrac{3}{2}\pi$, $\dfrac{7}{2}\pi$, that is, **y takes the minimum value $-1 - \dfrac{\sqrt{3}}{2}$ when $\theta = \dfrac{7}{12}\pi$, $\dfrac{19}{12}\pi$** .

(2) If $t = \sin\theta - \cos\theta$, then $t^2 = \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta = 1 - \sin 2\theta$, so $\sin 2\theta = 1 - t^2$.

Therefore, $y = \sin 2\theta - 2\sin\theta + 2\cos\theta = \sin 2\theta - 2(\sin\theta - \cos\theta) = (1 - t^2) - 2t = -t^2 - 2t + 1$

$= -(t+1)^2 + 1 + 1 = -(t+1)^2 + 2$.



Where $t = \sin\theta - \cos\theta = \sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right)$ ……①,

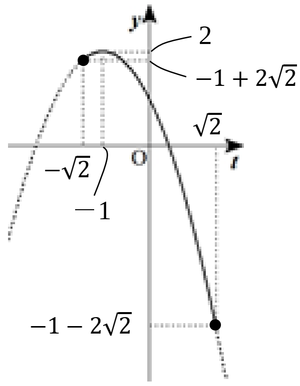
and since $0 \leqq \theta < 2\pi, -\frac{\pi}{4} \leqq \theta - \frac{\pi}{4} < \frac{7}{4}\pi$ ……②,

$-\sqrt{2} \leqq t \leqq \sqrt{2}$.

Therefore, from the figure on the right,

y has a maximum value of 2 when $t = -1$

and a minimum value of $-1 - 2\sqrt{2}$ when $t = \sqrt{2}$.



When $t = -1, \sin\left(\theta - \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ from ① .

Solving in the range of ② yields $\theta - \frac{\pi}{4} = -\frac{\pi}{4}, \ \frac{5}{4}\pi$.

That is, $\theta = 0, \ \frac{3}{2}\pi$.

When $t = \sqrt{2}, \sin\left(\theta - \frac{\pi}{4}\right) = 1$ from ① .

Solving in the range of ② yields $\theta - \frac{\pi}{4} = \frac{\pi}{2}$. That is, $\theta = \frac{3}{4}\pi$.

From the above, **the maximum value is 2 when $\boldsymbol{\theta = 0}$ and $\boldsymbol{\frac{3}{2}\pi}$,**

and the minimum value is $\boldsymbol{-1 - 2\sqrt{2}}$ when $\boldsymbol{\theta = \frac{3}{4}\pi}$.