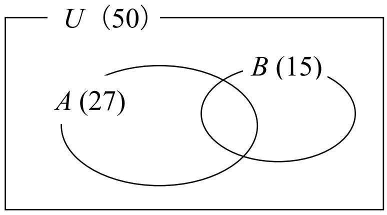
Number of cases

1

Among the 50 participants, 27 preferred coffee, 15 preferred tea, and 12 preferred neither coffee nor tea.
How many people like both coffee and tea?

solution



Let U be the whole set, and

let A be the set of those who like coffee and

B be the set of those who like tea.

At this time, $n(U)=50$, $n(A)=27$, $n(B)=15$, $n(\overline{A\cup B})=12$.

Since the number of cases to be sought is $n(A\cap B)$,

let this be x people,

$$n(A\cup B)=n(A)+n(B)-n(A\cap B)=27+15-x\ .$$

Also, $n(A\cup B)=n(U)-n(\overline{A\cup B})=50-12$.

Therefore, $50-12=27+15-x$.

Thus, the number of people sought is $x=$ **4（people）** .

2

(1) How many ways are there to pay 200 yen using 100 yen, 50 yen, and 10 yen coins?
However, there shall be a sufficient number of each coins, and there may be coins that are not used.

(2) There are 3 modes of transportation to get from prefecture A to prefecture B: bus, train, and airplane.
How many ways are there to go from prefecture A to prefecture B and back?
However, the same means of transportation may be used for the return trip.

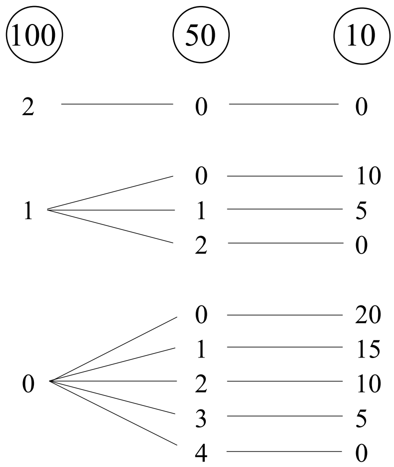
solution

(1) The method of paying 200 yen is shown in the tree diagram on the right.

(ⅰ) When two 100 yen coins are used
From the diagram on the right, it is 1 way.

(ⅱ) When there is one 100 yen coin
From the diagram on the right, there are 3 ways.

(ⅲ) When there are zero 100 yen coins
From the diagram on the right, there are 5 ways.



The numbers represent the number of coins used.

> The case classification can be reduced by starting from those with the largest amounts.

Since cases (ⅰ) - (ⅲ) do not occur simultaneously,
the number of cases we seek is

$1+3+5=$ **9 (ways)**.

(2) The total number of choices sought is

$3\times3=$ **9 (ways)**

by the law of product.

3

How many positive divisors of 60 are there in total? Also, find the sum of the divisors.

solution

$60=2^2\times3\times5$

From this, all positive divisors of 60 appear in the terms expanding $(2^0+2^1+2^2)(3^0+3^1)(5^0+5^1)$.

Therefore, the number of positive divisors we seek is $(2+1)\times(1+1)\times(1+1)=$ **12（pieces）**.

The sum of the divisors is $(1+2^1+2^2)(1+3^1)(1+5^1)=(1+2+4)(1+3)(1+5)=7\times4\times6=$ **168** .

4

(1) How many 3-digit integers can be obtained from the 7 integers 1, 2, 3, 4, 5, 6, and 7 by taking 3 different integers and arranging them in a row? How many of these integers are odd?

(2) How many 4-digit integers in total can be formed when four different integers are taken from the 6 integers 0, 1, 2, 3, 4, and 5 and arranged in a row?

(3) When 6 people, 3 boys and 3 girls, line up in a row, how many ways are there for the 3 girls to be next to each other? Also, how many ways are there in which the boys are at both ends of the line?

solution

(1) The number of 3-digit integers that can be formed by taking 3 different integers from the 7 integers and arranging them is $_7P_3 = 7 \times 6 \times 5 = \mathbf{210}$ **(pieces)**.

The odd numbers among these numbers are 1, 3, 5, or 7 in the first place, and there are 4 ways.

For each of them, the hundred and ten places are permutations that take 2 out of the remaining 6, so there are $_6P_2$ permutations.

Therefore, the number of pieces sought is $4 \times {_6P_2} = 4 \times 6 \times 5 = \mathbf{120}$ **(pieces)**.

(2) The thousandths place is taken from 1 to 5 except 0, so there are 5 ways.

For each of them, the hundred, ten, and one places are permutations that take 3 out of the remaining 5, including 0, so there are $_5P_3$ ways.

Therefore, the number of pieces sought is $5 \times {_5P_3} = 5 \times 5 \times 4 \times 3 = \mathbf{300}$ **(pieces)**.

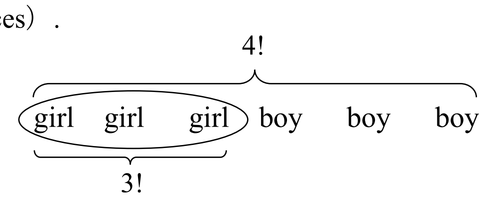
Alternative solution

The total number of permutations that can be arranged by taking 4 out of 6 integers from 0 to 5 is $_6P_4 = 6 \times 5 \times 4 \times 3 = 360$ (pieces).

Of these, $_5P_3 = 5 \times 4 \times 3 = 60$ (pieces) have a first integer of 0.

Therefore, the number of pieces sought is $360 - 60 = 300$ (pieces).

(3) 3 girls are considered as one group.

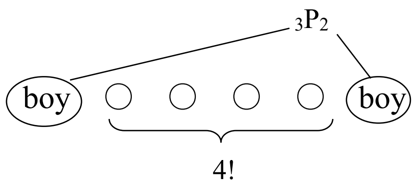


One group of three girls and 3 boys are lined up in 4! ways.

3 girls are lined up in 3! ways.

Therefore, the required sequence is

$$4! \times 3! = 4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1 = \mathbf{144} \text{ (ways)}.$$



In addition, there are $_3P_2$ ways for the boys to line up at both ends.

The remaining 4 people can be lined up in 4! ways.

Therefore, the required sequence is

$$_3P_2 \times 4! = 3 \times 2 \times 4 \times 3 \times 2 \times 1 = \mathbf{144} \text{ (ways)}.$$

5

(1) How many ways are there to arrange 7 different beads in the shape of a circle?

(2) How many ways are there to make a loop by threading 7 different beads?

(3) How many ways are there to paint each face of the cube using all 6 different colors?

However, the painting methods that match by rotating the cube are considered the same.

solution

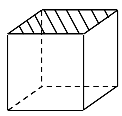
(1) There are $\frac{{}_7P_7}{7} = (7-1)! = 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = \mathbf{720}$ **(ways)**

to arrange 7 different beads in the shape of a circle.

(2) The loop threaded through the beads is $\frac{(7-1)!}{2} = \mathbf{360}$ **(ways)**,

since it is the same when turned inside out.

(3) First, fix the color of the top surface.



In this case, the color of the underside is the remaining color, in 5 ways.

For each of them, there are (4－1)! ways to paint the sides,

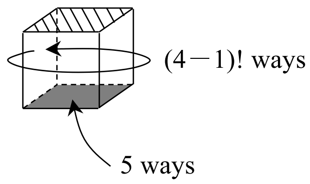
since they are 4 different circular permutations.

Therefore, there are

$$5 \times (4-1)! = 5 \times 3! = 5 \times 3 \times 2 \times 1$$

$$= \mathbf{30} \text{ (ways)}$$

to find it.



6

(1) How many 3-digit integers can be formed using the 4 different numbers 0, 1, 2, and 3?
However, the same number may be used repeatedly.

(2) When dividing 6 people into 2 rooms A and B, how many different ways are there to divide them so that every room has at least 1 person?

solution

(1) There are 3 numbers that can be used for the hundreds place: 1, 2, and 3.
There are 4 numbers that can be used for the tens and ones places: 0, 1, 2, and 3, respectively.
Therefore, the number of pieces sought is $3\times4^2=$ **48（pieces）**.

(2) Supposing that there is a possibility of a vacant room, there are

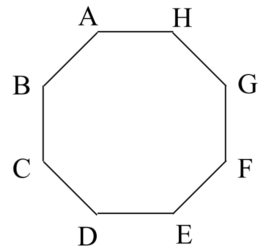
$$2^6=64\text{（ways）}$$

to divide the 6 people into 2 rooms, A and B.
Of these, there are 2 ways to enter only one of A and B.
Therefore, the division is $64-2=$ **62（ways）**.

7

(1) How many ways are there to choose 6 from 9?

(2) How many triangles can be formed in total by selecting the 3 vertices of a regular octagon ABCDEFGH? How many triangles can be made that do not share a side with a regular octagon?



solution

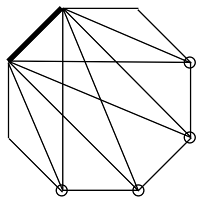
(1) ${}_9C_6 = {}_9C_3 = \dfrac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1} = \mathbf{84}$ **(ways)**

(2) There are ${}_8C_3 = \dfrac{8\cdot 7\cdot 6}{3\cdot 2\cdot 1} = \mathbf{56}$ **(pieces)** triangles made by selecting 3 vertices from the 8 vertices.

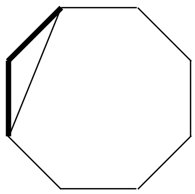
Also, the number of triangles that share only one side with a regular octagon is

$(8-4)\times 8 = 32$ (pieces),

since for each side we only need to choose a vertex excluding both ends of the side and both neighbors.



The number of triangles that share two sides with a regular octagon is 8, since they are triangles formed by two adjacent sides.



From the above, the number of triangles we seek is

$56-(32+8) = \mathbf{16}$ **(pieces)**.

8

How many ways are there to divide the 8 people as follows?

(1) Divide the group into four pairs of two each, A, B, C, and D.

(2) Divide the group into four pairs of two people each.

(3) Divide the group into three groups of three, three, and two.

solution

(1) There are ${}_8C_2$ ways to choose the two people to put in A.

There are ${}_6C_2$ ways to choose the 2 people to be placed in B from the remaining 6.

There are ${}_4C_2$ ways to choose the 2 people to be placed in C from the remaining 4.

Since the remaining 2 people are placed in D, there are

$${}_8C_2\times{}_6C_2\times{}_4C_2=\frac{8\cdot7}{2\cdot1}\times\frac{6\cdot5}{2\cdot1}\times\frac{4\cdot3}{2\cdot1}=\mathbf{2520}\ \textbf{(ways)}$$

of dividing the number of people to be sought.

(2) In (1), if we eliminate the distinction between A, B, C, and D, we get 4! of the same thing each, so there are

$$\frac{{}_8C_2\times{}_6C_2\times{}_4C_2}{4!}=\frac{2520}{24}=\mathbf{105}\ \textbf{(ways)}$$

to find the division.

(3) There are

$${}_8C_3\times{}_5C_3={}_8C_3\times{}_5C_2=\frac{8\cdot7\cdot6}{3\cdot2\cdot1}\times\frac{5\cdot4}{2\cdot1}=560\ \text{(ways)}$$

to divide 3, 3, and 2 into 3 pairs, A, B, and C.

If we eliminate the distinction between A and B, we can obtain the same thing in 2! ways each, so there are

$$\frac{{}_8C_3\times{}_5C_3}{2!}=\frac{560}{2}=\mathbf{280}\ \textbf{(ways)}$$

to find the division.

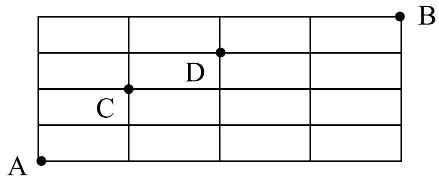
9

In the diagram on the right, how many different paths can be taken to get from point A to point B in the shortest way in the following cases?

(1) All Directions

(2) Directions through point C

(3) Directions that do not pass through both points C and D



solution

(1) One section up is indicated by ↑, and one section to the right is indicated by → . The shortest path is represented by 4 ↑ and 4 → permutations, so there are $\frac{8!}{4!4!}=\frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1\cdot4\cdot3\cdot2\cdot1}=$**70（ways）**.

(2) The shortest path from point A to point C is represented by a permutation of 2 ↑ and 1 →, so there are $\frac{3!}{2!1!}$ ways.

The shortest path from point C to point B is represented by a permutation of 2 ↑ and 3 →, so there are $\frac{5!}{2!3!}$ ways.

Therefore, there are $\frac{3!}{2!1!}\times\frac{5!}{2!3!}=\frac{3\cdot2\cdot1}{2\cdot1\cdot1}\times\frac{5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot3\cdot2\cdot1}=3\times10=$**30（ways）** to seek.

(3) (route not passing through both points C and D) ＝ (all routes) － (routes passing through points C or D) .

Where (route through point C or D) ＝ (route through point C) ＋ (route through point D)

－ (route through points C and D) .

First, find the route through point D.

There are $\frac{5!}{3!2!}$ ways from point A to point D : 3 ↑ and 2 → .

There are $\frac{3!}{1!2!}$ ways from point D to point B : 1 ↑ and 2 → .

Therefore, there are $\frac{5!}{3!2!}\times\frac{3!}{1!2!}=\frac{5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot2\cdot1}\times\frac{3\cdot2\cdot1}{1\cdot2\cdot1}=10\times3=30$ (ways) .

Next, find the route through points C and D.

There are $\frac{3!}{2!1!}$ ways from point A to point C : 2 ↑ and 1 → .

There are $\frac{2!}{1!1!}$ ways from point C to point D : 1 ↑ and 1 → .

There are $\frac{3!}{1!2!}$ ways from point D to point B : 1 ↑ and 2 → .

Therefore, there are $\frac{3!}{2!1!}\times\frac{2!}{1!1!}\times\frac{3!}{1!2!}=\frac{3\cdot2\cdot1}{2\cdot1\cdot1}\times\frac{2\cdot1}{1\cdot1}\times\frac{3\cdot2\cdot1}{1\cdot2\cdot1}=3\times2\times3=18$ (ways) .

From the above, there are $70-(30+30-18)=$**28 (ways)** to seek.

10

(1) How many pairs of integers (x, y, z) are there in total satisfying $x+y+z=8$, $x \geqq 0$, $y \geqq 0$, $z \geqq 0$?

(2) How many pairs of natural numbers (l, m, n) satisfy $l+m+n=8$ in total?

solution

(1) It is the total number of combinations of 3 different types, allowing for duplicates and taking 8 of them, so

$${}_3H_8 = {}_{3+8-1}C_8 = {}_{10}C_2 = \frac{10 \cdot 9}{2 \cdot 1} = \mathbf{45}\ \textbf{(pairs)}.$$

Alternative solution

Consider 8 ○ and 2 partitions ｜, e.g.

○○｜○○○｜○○○ is (x, y, z)=(2, 3, 3),

｜○○○○○｜○○○ represents (x, y, z)=(0, 5, 3).

The total number of pairs of (x, y, z) to be obtained is

$${}_{10}C_2 = \frac{10 \cdot 9}{2 \cdot 1} = 45\ \text{(pairs)}.$$

(2) Since l, m, and n are natural numbers, $l \geqq 1$, $m \geqq 1$, and $n \geqq 1$, they cannot be 0. Then, let

$l-1=X$, $m-1=Y$, and $n-1=Z$

and consider integers $X \geqq 0$, $Y \geqq 0$, and $Z \geqq 0$ that are greater than or equal to 0. Substituting $l=X+1$, $m=Y+1$, and $n=Z+1$ into the given equations, we obtain

$(X+1)+(Y+1)+(Z+1)=8$. That is, $X+Y+Z=5$, $X \geqq 0$, $Y \geqq 0$, $Z \geqq 0$.

It is the total number of combinations of 5 pieces taken from 3 different types, allowing for duplicates, so

$${}_3H_5 = {}_{3+5-1}C_5 = {}_7C_2 = \frac{7 \cdot 6}{2 \cdot 1} = \mathbf{21}\ \textbf{(pairs)}.$$

Alternative solution

When 8 ○ are arranged in a row, consider putting 2 partitions ｜ in the 7 places between them.

For example, ○○｜○○○｜○○○ represents (l, m, n)=(2, 3, 3),

so the total number of pairs of (l, m, n) to be obtained is

$${}_7C_2 = \frac{7 \cdot 6}{2 \cdot 1} = 21\ \text{(pairs)}.$$

Study 1

When 6 people, A, B, C, D, E, and F, line up in a row, how many ways are there to line up so that A, B, and C are not next to each other?

solution

First, line up D, E, and F.

1 Ⓓ 2 Ⓔ 3 Ⓕ 4

Then, if A, B, and C are placed between or at the two ends, 1 through 4,
A, B, and C will not be adjacent to each other.

There are 3! ways to line up D, E, and F.

The sequence of A, B, and C in 1 - 4 is ${}_4P_3$ ways.

Therefore, the required sequence is $3! \times {}_4P_3 = 3 \cdot 2 \cdot 1 \times 4 \cdot 3 \cdot 2 = \mathbf{144}$ **(ways)**.

Study 2

When the eight letters M, O, U, N, T, A, I, N are arranged in a horizontal row, how many ways can O, U, A, I be arranged in this order?

solution

O, U, A, and I are considered to be the same, i.e., 4 □.

□ : 4, M : 1, N : 2, and T : 1 are arranged in a horizontal row,

and O, U, A, and I can be placed in the 4 □ from the left to the right.

Therefore, the required sequence is

$$\frac{8!}{4!1!2!1!}=\frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{4\cdot 3\cdot 2\cdot 1\cdot 1\cdot 2\cdot 1\cdot 1}=\mathbf{840}\ \textbf{(ways)}\ .$$

For example, the ordering

□N□□T□M N

corresponds to the ordering

O N U A T I M N

with O, U, A, and I from the left.