Real number, Linear inequality

1

(1) Express the following fractions in decimals.

① $\frac{1}{12}$　　② $\frac{3}{16}$

(2) Express the following circular decimals in fractions.

① $0.\dot{8}$　　② $1.\dot{2}\dot{3}$

solution

(1) ① $\frac{1}{12}=0.08333\cdots\cdots=\mathbf{0.08\dot{3}}$

② $\frac{3}{16}=\mathbf{0.1875}$

①
$$\begin{array}{r} 0.083 \\ 12\,\overline{)\,1.0} \\ \underline{96} \\ 4 \\ \underline{36} \\ 4 \end{array}$$

②
$$\begin{array}{r} 0.1875 \\ 16\,\overline{)\,3.0} \\ \underline{16} \\ 140 \\ \underline{128} \\ 120 \\ \underline{112} \\ 80 \\ \underline{80} \\ 0 \end{array}$$

(2) ① If we leave it as $x=0.\dot{8}$, then it is $10x=8.\dot{8}$,

and so $10x-x=8$.

Therefore, $x=\mathbf{\frac{8}{9}}$.

$$\begin{array}{rl} 10x & =8.888\cdots\cdots \\ -)\quad x & =0.888\cdots\cdots \\ \hline 9x & =8 \end{array}$$

② If we leave it as $x=1.\dot{2}\dot{3}$, then it is $100x=123.\dot{2}\dot{3}$,

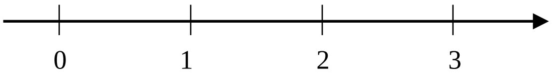
and so $100x-x=122$.

Therefore, $x=\mathbf{\frac{122}{99}}$.

$$\begin{array}{rl} 100x & =123.2323\cdots\cdots \\ -)\quad x & =1.2323\cdots\cdots \\ \hline 99x & =122 \end{array}$$

2

(1) Take points $P\left(\frac{7}{4}\right)$ and $Q\left(\sqrt{3}\right)$ on the right number line.



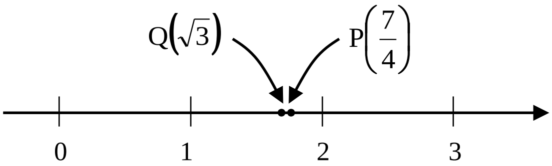
(2) Find the following values.

① $\left|-\frac{1}{2}\right|$ ② $|\sqrt{2}-\sqrt{3}|$ ③ $|1|-|-2|$

(3) Find the value of $|2+\sqrt{5}||2-\sqrt{5}|$.

solution

(1)



(2) ① $\left|-\frac{1}{2}\right|=\mathbf{\frac{1}{2}}$

② $\sqrt{2}-\sqrt{3}<0$, so it is $|\sqrt{2}-\sqrt{3}|=\mathbf{-\sqrt{2}+\sqrt{3}}$.

③ $|1|-|-2|=1-2=\mathbf{-1}$

(3) $|2+\sqrt{5}||2-\sqrt{5}|=|(2+\sqrt{5})(2-\sqrt{5})|$

$=|4-5|=|-1|=\mathbf{1}$

3

(1) Find the following values.

① $(-\sqrt{5})^2$ ② $-\sqrt{3^2}$

(2) Simplify the following expressions involving square roots.

① $\sqrt{27}$ ② $\sqrt{6}\sqrt{15}$ ③ $\dfrac{\sqrt{50}}{\sqrt{2}}$ ④ $\sqrt{0.12}$

solution

(1) ① $(-\sqrt{5})^2=\mathbf{5}$

② $-\sqrt{3^2}=-|3|=\mathbf{-3}$

(2) ① $\sqrt{27}=\sqrt{3^2\cdot 3}=\mathbf{3\sqrt{3}}$

② $\sqrt{6}\sqrt{15}=\sqrt{6\cdot 15}=\sqrt{3^2\cdot 10}=\mathbf{3\sqrt{10}}$

③ $\dfrac{\sqrt{50}}{\sqrt{2}}=\sqrt{\dfrac{50}{2}}=\sqrt{25}=\sqrt{5^2}=\mathbf{5}$

④ $\sqrt{0.12}=\sqrt{\dfrac{12}{100}}=\dfrac{\sqrt{12}}{\sqrt{100}}=\dfrac{\sqrt{2^2\cdot 3}}{\sqrt{10^2}}=\dfrac{2\sqrt{3}}{10}=\mathbf{\dfrac{\sqrt{3}}{5}}$

4

(1) Calculate the following expressions involving square roots.

① $\sqrt{54}+\sqrt{96}$ ② $(3-\sqrt{6})(3+\sqrt{6})$

③ $(2-\sqrt{2})^2$ ④ $(1+2\sqrt{3})(3-\sqrt{3})$

(2) Rationalize the denominator of the following expressions involving square roots.

① $\dfrac{2}{\sqrt{3}}$ ② $\dfrac{1-\sqrt{6}}{\sqrt{2}}$ ③ $\dfrac{1}{\sqrt{2}-\sqrt{5}}$ ④ $\dfrac{3-2\sqrt{2}}{3+2\sqrt{2}}$

solution

(1) ① $\sqrt{54}+\sqrt{96}=\sqrt{3^2\cdot 6}+\sqrt{4^2\cdot 6}=3\sqrt{6}+4\sqrt{6}=\mathbf{7\sqrt{6}}$

② $(3-\sqrt{6})(3+\sqrt{6})=3^2-(\sqrt{6})^2=9-6=\mathbf{3}$

③ $(2-\sqrt{2})^2=2^2-2\cdot 2\cdot\sqrt{2}+(\sqrt{2})^2=4-4\sqrt{2}+2=\mathbf{6-4\sqrt{2}}$

④ $(1+2\sqrt{3})(3-\sqrt{3})=1\cdot 3+\{1\cdot(-1)+2\cdot 3\}\sqrt{3}+\{2\cdot(-1)\}(\sqrt{3})^2$

$=3+5\sqrt{3}-2\cdot 3=\mathbf{-3+5\sqrt{3}}$

Alternative solution

$(1+2\sqrt{3})(3-\sqrt{3})=1\cdot 3+1\cdot(-\sqrt{3})+2\sqrt{3}\cdot 3+2\sqrt{3}\cdot(-\sqrt{3})$

$=3-\sqrt{3}+6\sqrt{3}-6=-3+5\sqrt{3}$

(2) ① $\dfrac{2}{\sqrt{3}}=\dfrac{2\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}}=\mathbf{\dfrac{2\sqrt{3}}{3}}$

② $\dfrac{1-\sqrt{6}}{\sqrt{2}}=\dfrac{(1-\sqrt{6})\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}=\mathbf{\dfrac{\sqrt{2}-2\sqrt{3}}{2}}$

③ $\dfrac{1}{\sqrt{2}-\sqrt{5}}=\dfrac{\sqrt{2}+\sqrt{5}}{(\sqrt{2}-\sqrt{5})(\sqrt{2}+\sqrt{5})}=\dfrac{\sqrt{2}+\sqrt{5}}{(\sqrt{2})^2-(\sqrt{5})^2}=\dfrac{\sqrt{2}+\sqrt{5}}{2-5}=\mathbf{-\dfrac{\sqrt{2}+\sqrt{5}}{3}}$

④ $\dfrac{3-2\sqrt{2}}{3+2\sqrt{2}}=\dfrac{(3-2\sqrt{2})^2}{(3+2\sqrt{2})(3-2\sqrt{2})}=\dfrac{3^2-2\cdot 3\cdot 2\sqrt{2}+(2\sqrt{2})^2}{3^2-(2\sqrt{2})^2}=\dfrac{9-12\sqrt{2}+8}{9-8}=\mathbf{17-12\sqrt{2}}$

5

If $a<b$, put an inequality sign in the following blanks $\square$.

(1) $a+5 \square b+5$ (2) $3a \square 3b$ (3) $-\frac{1}{4}a \square -\frac{1}{4}b$

(4) $\frac{a}{2}-5 \square \frac{b}{2}-5$ (5) $-2a+6 \square -2b+6$

solution

(1) $a+5 \boxed{<} b+5$

(2) $3a \boxed{<} 3b$

(3) $-\frac{1}{4}a \boxed{>} -\frac{1}{4}b$

(4) When $a<b$, $\frac{a}{2}<\frac{b}{2}$, and subtracting 5 from both of these sides does not change the direction of the inequality $\frac{a}{2}-5 \boxed{<} \frac{b}{2}-5$.

(5) When $a<b$, $-2a>-2b$, and adding 6 to both sides does not change the direction of the inequality $-2a+6 \boxed{>} -2b+6$.

6

Solve the following inequalities.

(1) $x+2 \leqq -3$ (2) $-3x > -9$ (3) $2x-5 \geqq -1$ (4) $-5x-3 < 7$

(5) $2x+3 \geqq -2x-5$ (6) $x+4 \leqq 10+4x$ (7) $2(3x-1) > 3(4x+5)+1$ (8) $\frac{x+8}{6} < \frac{x}{4}+1$

solution

(1) $x+2 \leqq -3$.

Transposing, we get $\boldsymbol{x \leqq -5}$.

(2) $-3x > -9$.

Dividing both sides by -3, we get $\boldsymbol{x < 3}$.

(3) $2x-5 \geqq -1$.

Transposing, we have $2x \geqq 4$.

Dividing both sides by 2, we get $\boldsymbol{x \geqq 2}$.

(4) $-5x-3 < 7$.

Transposing, we have $-5x < 10$.

Dividing both sides by -5, we get $\boldsymbol{x > -2}$.

(5) $2x+3 \geqq -2x-5$.

Transposing, we have $2x+2x \geqq -5-3$. To summarize, we have $4x \geqq -8$.

Dividing both sides by 4, we get $\boldsymbol{x \geqq -2}$.

(6) $x+4 \leqq 10+4x$.

Transposing, we have $x-4x \leqq 10-4$. To summarize, we have $-3x \leqq 6$.

Dividing both sides by -3, we get $\boldsymbol{x \geqq -2}$.

(7) $2(3x-1) > 3(4x+5)+1$.

Remove the parentheses, we have $6x-2 > 12x+15+1$.

Transposing, we have $6x-12x > 15+1+2$. To summarize, we have $-6x > 18$.

Dividing both sides by -6, we get $\boldsymbol{x < -3}$.

(8) $\frac{x+8}{6} < \frac{x}{4}+1$.

Multiplying both sides by 12, we have $2(x+8) < 3x+12$.

Remove the parentheses, we have $2x+16 < 3x+12$.

Transposing, we have $2x-3x < 12-16$. To summarize, we have $-x < -4$.

Dividing both sides by -1, we get $\boldsymbol{x > 4}$.

7

(1) Solve the following simultaneous inequalities.

① $\begin{cases} x+6 \leqq 4x \\ 1-3x<15-5x \end{cases}$ ② $\begin{cases} 2(x+6) \leqq 3(4-x) \\ 0.7x+0.5<x+2 \end{cases}$

(2) Solve inequality $x-2<-\frac{1}{2}x+1<-3x-4$.

solution

(1) ① $\begin{cases} x+6 \leqq 4x & \cdots\cdots \text{(i)} \\ 1-3x<15-5x & \cdots\cdots \text{(ii)} \end{cases}$.

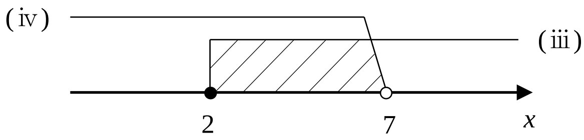
From (i) , we have $-3x \leqq -6$.

Therefore , $x \geqq 2$ ……(iii) .

From (ii) , we have $-3x+5x<15-1$.

Therefore , $2x<14$.

Solving this yields $x<7$ ……(iv) .



Finding the common range of (iii) and (iv), we get $\mathbf{2 \leqq x < 7}$.

② $\begin{cases} 2(x+6) \leqq 3(4-x) & \cdots\cdots \text{(i)} \\ 0.7x+0.5<x+2 & \cdots\cdots \text{(ii)} \end{cases}$.

From (i) , we have $2x+12 \leqq 12-3x$.

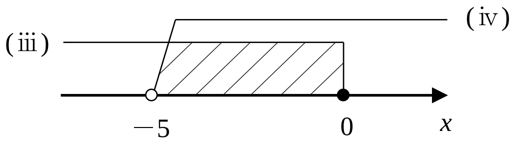
Therefore , $5x \leqq 0$.

Solving this yields $x \leqq 0$ ……(iii) .

From (ii) , we have $7x+5<10x+20$.

Therefore , $-3x<15$.

Solving this yields $x>-5$ ……(iv) .



Finding the common range of (iii) and (iv), we get $\mathbf{-5 < x \leqq 0}$.

(2) $\begin{cases} x-2<-\frac{1}{2}x+1 & \cdots\cdots \text{(i)} \\ -\frac{1}{2}x+1<-3x-4 & \cdots\cdots \text{(ii)} \end{cases}$.

From (i) , we have $2x-4<-x+2$.

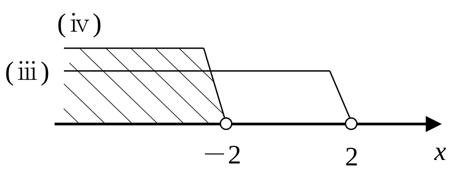
Therefore , $3x<6$.

Solving this yields $x<2$ ……(iii) .

From (ii) , we have $-x+2<-6x-8$.

Therefore , $5x<-10$.

Solving this yields $x<-2$ ……(iv) .



Finding the common range of (iii) and (iv), we get $\mathbf{x<-2}$.

8

Solve the following equations and inequalities.

(1) $|2x-5|=3$ (2) $|x+4|\leqq 6$ (3) $|3x-1|>2$

solution

(1) From $|2x-5|=3$, we have $2x-5=\pm 3$.

Therefore , $2x=5\pm 3$. Thus , $\boldsymbol{x=4,\ 1}$.

(2) From $|x+4|\leqq 6$, we have $-6\leqq x+4\leqq 6$.

Thus , $\boldsymbol{-10\leqq x\leqq 2}$.

(3) From $|3x-1|>2$, we have $3x-1<-2$, $2<3x-1$.

Therefore , $3x<-1$, $3<3x$. Thus , $\boldsymbol{x<-\frac{1}{3},\ 1<x}$.

9

Solve the following equations and inequalities.

(1) $|x| = 3x - 2$ (2) $|3x-2| \geqq x+2$

solution

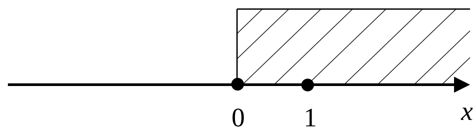
(1) $|x| = 3x - 2$

(ⅰ) $x \geqq 0$, then

$$x = 3x - 2.$$

Solving this yields $x = 1$.

This satisfies $x \geqq 0$.

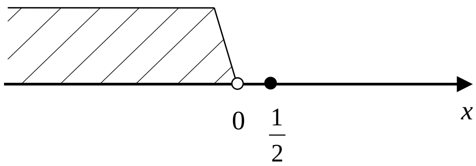


(ⅱ) $x < 0$, then

$$-x = 3x - 2.$$

Solving this yields $x = \frac{1}{2}$.

This does not satisfy $x < 0$.



From (ⅰ) and (ⅱ), the solution of the equation is $\boldsymbol{x = 1}$.

(2) $|3x-2| \geqq x+2$

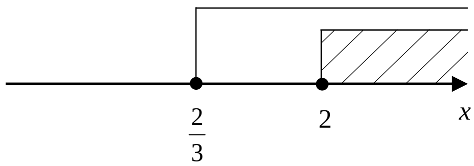
(ⅰ) $3x - 2 \geqq 0$, i.e., $x \geqq \frac{2}{3}$, then

$$3x - 2 \geqq x + 2.$$

Solving this yields $x \geqq 2$.

The common range between this and $x \geqq \frac{2}{3}$ is

$$x \geqq 2 \quad \cdots\cdots①.$$



(ⅱ) $3x - 2 < 0$, i.e., $x < \frac{2}{3}$, then

$$-(3x-2) \geqq x + 2.$$

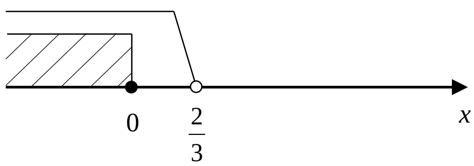
$$-3x + 2 \geqq x + 2.$$

Therefore, $-4x \geqq 0$.

Solving this yields $x \leqq 0$.

The common range between this and $x < \frac{2}{3}$ is

$$x \leqq 0 \quad \cdots\cdots②.$$



From (ⅰ) and (ⅱ), the solution to the inequality is $\boldsymbol{x \leqq 0,\ 2 \leqq x}$,

combining ① and ②.

Study 1

If the integer part of $2\sqrt{3}$ is a and the decimal part is b, find the value of the following expressions.

(1) a (2) b (3) $\dfrac{a}{b}$

solution

(1) From $2\sqrt{3}=\sqrt{12}$, $\sqrt{9}<\sqrt{12}<\sqrt{16}$. Therefore, $3<2\sqrt{3}<4$.

Thus, the integer part a of $2\sqrt{3}$ is $a=\mathbf{3}$.

(2) $b=\mathbf{2\sqrt{3}-3}$

(3) $\dfrac{a}{b}=\dfrac{3}{2\sqrt{3}-3}=\dfrac{3(2\sqrt{3}+3)}{(2\sqrt{3}-3)(2\sqrt{3}+3)}=\dfrac{3(2\sqrt{3}+3)}{12-9}$

$=\mathbf{2\sqrt{3}+3}$

Study 2

Put the following expressions involving square roots into simple form.

(1) $\sqrt{3+2\sqrt{2}}$ (2) $\sqrt{7-2\sqrt{6}}$ (3) $\sqrt{7-4\sqrt{3}}$ (4) $\sqrt{2+\sqrt{3}}$

solution

(1) $\sqrt{3+2\sqrt{2}}=\sqrt{(2+1)+2\sqrt{2\cdot 1}}=\sqrt{2}+\sqrt{1}=\mathbf{\sqrt{2}+1}$

(2) $\sqrt{7-2\sqrt{6}}=\sqrt{(6+1)-2\sqrt{6\cdot 1}}=\sqrt{6}-\sqrt{1}=\mathbf{\sqrt{6}-1}$

(3) $\sqrt{7-4\sqrt{3}}=\sqrt{7-2\sqrt{12}}=\sqrt{(4+3)-2\sqrt{4\cdot 3}}=\sqrt{4}-\sqrt{3}=\mathbf{2-\sqrt{3}}$

(4) $\sqrt{2+\sqrt{3}}=\sqrt{\frac{4+2\sqrt{3}}{2}}=\sqrt{\frac{(3+1)+2\sqrt{3\cdot 1}}{2}}=\frac{\sqrt{3}+\sqrt{1}}{\sqrt{2}}=\mathbf{\frac{\sqrt{6}+\sqrt{2}}{2}}$

Study 3

$x=\dfrac{3-\sqrt{6}}{3+\sqrt{6}}$, $y=\dfrac{3+\sqrt{6}}{3-\sqrt{6}}$, find the value of the following expressions.

(1) $x+y$, xy (2) x^2+y^2 (3) x^3+y^3

solution

(1) $x+y=\dfrac{3-\sqrt{6}}{3+\sqrt{6}}+\dfrac{3+\sqrt{6}}{3-\sqrt{6}}=\dfrac{(3-\sqrt{6})^2+(3+\sqrt{6})^2}{(3+\sqrt{6})(3-\sqrt{6})}=\dfrac{9-6\sqrt{6}+6+9+6\sqrt{6}+6}{9-6}=\mathbf{10}$

$xy=\dfrac{3-\sqrt{6}}{3+\sqrt{6}}\cdot\dfrac{3+\sqrt{6}}{3-\sqrt{6}}=\mathbf{1}$

(2) $x^2+y^2=(x+y)^2-2xy=10^2-2\cdot 1$

$=\mathbf{98}$

(3) $x^3+y^3=(x+y)^3-3xy(x+y)=10^3-3\cdot 1\cdot 10=1000-30$

$=\mathbf{970}$